Structured State Space Realizations for SLS Distributed Controllers

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Abstract— **In recent work the system level synthesis (SLS) paradigm has been shown to provide a truly scalable method for synthesizing distributed feedback controllers. Moreover, the resulting synthesis problem is convex. In this paper we provide minimal state space realizations for both the state and output feedback controllers. It is also shown that (for fixed n) the state dimension of the controllers grows linearly with FIR filter horizon length – in both cases, we show that if the underlying transfer matrices are structured, so is the corresponding state-space realization. For the $\mathcal{H}_2$ state feedback case a simple decomposition technique reduces the synthesis problem to solving a set of lower dimensional LQR problems that can be solved in parallel.**

I. Preliminaries & Notation

We use standard definitions of the Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$ and denote their restriction to the set of real-rational proper transfer matrices by $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$ respectively. For a detailed desription of these spaces the reader is referred to [1]. Let G_i denote the i^{th} spectral component of a transfer function $\mathbf{G}$, i.e., $\mathbf{G}(z) = \sum_{i=0}^{\infty} \frac{1}{z^i} G_i$ for $|z| > 1$. Finally, we use $\mathcal{F}_T$ to denote the space of finite impulse response (FIR) transfer matrices with horizon T, i.e., $\mathcal{F}_T := \{\mathbf{G} \in \mathcal{RH}_\infty \,|\, \mathbf{G} = \sum_{i=0}^{T} \frac{1}{z^i} G_i\}$.

We will make frequent use of the following system operations in this paper. Consider the proper MIMO systems

$$\mathbf{G}_i = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad i \in \{1, 2\}$$

whose inverses are given by

$$\mathbf{G}_i^{-1} = \left[\begin{array}{c|c} A_i - B_i D_i^\dagger C_i & -B_i D_i^\dagger \\ \hline D_i^\dagger C_i & D_i^\dagger \end{array}\right]$$

where $D_i^\dagger$ denotes the right inverse of D_i (which is assumed to have full row rank). Note that if the system is strictly proper, then there is no state space formula for the inverse. The cascade connection of two systems such that $\mathbf{y} = \mathbf{G}_1\mathbf{G}_2\mathbf{u}$ has a realization

$$\mathbf{G}_1\mathbf{G}_2 = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right],$$

and a parallel negative connection giving $\mathbf{y} = (\mathbf{G}_1 - \mathbf{G}_2)\mathbf{u}$ has the realization

$$\mathbf{G}_1 - \mathbf{G}_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & -C_2 & D_1 - D_2 \end{array}\right].$$

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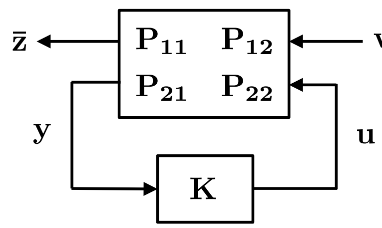

Fig. 1: Controller and plant interconnection for the control synthesis problem. The LFT $F(\mathbf{P}, \mathbf{K})$ is the closed loop map from $\mathbf{w}$ to $\bar{\mathbf{z}}$.

II. Introduction

A. Distributed Control

Consider the discrete, linear time invariant (LTI) system with state dimension n, n_u inputs, and n_o outputs:

$$x[k+1] = Ax[k] + B_1 w[k] + B_2 u[k] \tag{1a}$$
$$\bar{z}[k] = C_1 x[k] + D_{11} w[k] + D_{12} u[k] \tag{1b}$$
$$y[k] = C_2 x[k] + D_{21} w[k] + D_{22} u[k] \tag{1c}$$

which we compactly write as

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}.$$

In the frequency (z) domain the subsystems take the form $\mathbf{P}_{ij} = C_i(zI - A)^{-1}B_j + D_{ij}$.

Distributed control in the classical setting seeks to construct a controller $\mathbf{K}$ that solves

$$\begin{array}{rl} \text{minimize} & \|F(\mathbf{P}, \mathbf{K})\| \\ \text{subject to} & \mathbf{K} \text{ internally stabilizes } \mathbf{P} \\ & \mathbf{K} \in \mathcal{C} \end{array} \tag{2}$$

where $\|\cdot\|$ is an appropriate system norm, $\mathcal{C}$ is a subspace, and

$$F(\mathbf{P}, \mathbf{K}) \triangleq \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}$$

is the lower linear fractional transformation (LFT) of $\mathbf{P}$ and $\mathbf{K}$. The interconnection of plant and controller is shown in Figure 1. The subspace constraint $\mathbf{K} \in \mathcal{C}$ renders the distributed control problem non-convex in general. However, one of the centrepieces of decentralized control theory introduces the concept *quadratic invariance (QI)* which characterizes when (2) can be solved using convex programming [2].

The identification of QI as a useful condition for determining the tractability of a distributed optimal control problem led to an explosion of synthesis results in this area [3]–[11].

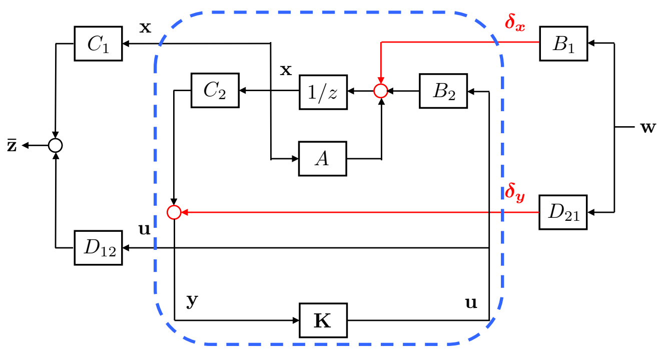

Fig. 2: Block diagram of the closed loop map that includes the plant state space matrices from (1). The key difference between the SLS formulation and the traditional approach is that the focus here is on the mapping from $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y})$ to $\mathbf{x}, \mathbf{u}$ rather than $\mathbf{w}$ to $\bar{\mathbf{z}}$.

These results showed that the robust and optimal control methods that proved so powerful for centralized systems could be ported to distributed settings. However, they also made clear that the synthesis and implementation of QI distributed optimal controllers did not scale gracefully with the size of the underlying cyber physical system. In particular, a QI distributed optimal controller is at least as expensive to compute as its centralized counterpart (c.f., the solutions presented in [3]–[11]), and can be more difficult to implement (c.f., the message passing implementation suggested in [11]). This limited scalability is what motivated the development of the System Level Synthesis (SLS) framework, which we briefly recall in the next subsection.

B. System Level Synthesis

System level synthesis (SLS) problems [12] define the broadest class of distributed, constrained optimal control problems that can be solved using convex optimization. The framework has been developed over the last few years to cover state feedback with delays [13], localized LQR and LQG control [14], [15], as well as filtering [16] and robustness [17]. The framework allows one to naturally handle communication and computational delay, sparsity, and locality constraints. Moreover, SLS problems have been shown to scale gracefully with system complexity, in particular, they enjoy $O(1)$ synthesis and implementation complexity relative to the dimension of the plant. For computational aspects of the SLS framework see [18].

For an LTI system with dynamics given by (1), we define the *system response* $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ to be the maps satisfying

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta_x} \\ \boldsymbol{\delta_y} \end{bmatrix}, \tag{3}$$

where $\boldsymbol{\delta_x} = B_1\mathbf{w}$ is the disturbance on the state vector, and $\boldsymbol{\delta_y} = D_{21}\mathbf{w}$ is the disturbance on the measurement. The closed loop map is shown in Figure 2 with the plant state space matrices included.

We say that a system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ is *stable and achievable* with respect to a plant $\mathbf{P}$ if there exists an internally stabilizing controller $\mathbf{K}$ such that the interconnection $F(\mathbf{P}, \mathbf{K})$ is consistent with (3). We make the following standard assumptions throughout this paper:

Assumption 1: The interconnection $F(\mathbf{P}, \mathbf{K})$ is well posed, i.e. the matrix $(I - D_{22}D_k)$ is invertible.

Assumption 2: Both the plant and the controller realizations are stabilizable and detectable; i.e., (A, B_2) and (A_k, B_k) are stabilizable, and (A, C_2) and (A_k, C_k) are detectable.

The following theorem characteizes all possible system responses a stabilizing controller can achieve.

Theorem 1 ([12]): For the output feedback problem with $D_{22} = 0$ in (1) the system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ transfer matrices from (3) are:

$$\begin{aligned} \mathbf{R} &= (zI - A - B_2\mathbf{K}C_2)^{-1} \\ \mathbf{M} &= \mathbf{K}C_2\mathbf{R} \\ \mathbf{N} &= \mathbf{R}B_2\mathbf{K} \\ \mathbf{L} &= \mathbf{K} + \mathbf{K}C_2\mathbf{R}B_2\mathbf{K}. \end{aligned} \tag{4}$$

and the following are true:

(a) The affine subspace described by:

$$\begin{bmatrix} zI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \tag{5a}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} zI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{5b}$$

$$\mathbf{R}, \mathbf{M}, \mathbf{N} \in \frac{1}{z}\mathcal{RH}_\infty, \quad \mathbf{L} \in \mathcal{RH}_\infty \tag{5c}$$

parameterizes all system responses (4) achievable by an internally stabilizing controller $\mathbf{K}$.

(b) For any transfer matrices $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ satisfying (5), the controller $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$ is internally stabilizing and achieves the desired response (4).

In [12], one implementation of the controller from the system response matrices $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ is given by

$$\begin{aligned} \boldsymbol{\beta} &= (I - z\mathbf{R})\boldsymbol{\beta} - \mathbf{N}\mathbf{y} \\ \mathbf{u} &= \mathbf{M}\boldsymbol{\beta} + \mathbf{L}\mathbf{y} \end{aligned} \tag{6}$$

where $\boldsymbol{\beta}$ is the controller's internal state. The similarities between this and the standard state space controllers defined by the matrices (A_K, B_K, C_K, D_K) given by

$$\begin{aligned} x_K[k+1] &= A_K x_K[k] + B_K y[k] \\ u[k] &= C_K x_K[k] + D_K y[k] \end{aligned} \tag{7}$$

are clear. The main difference however is that the controller matrices in the system level synthesis framework are dynamic, i.e. they contain transfer matrices. The contribution of this paper is to provide a classical realization (7) of the SLS controller (6).

III. State Space Realizations

In this section the main results of the paper are presented – structured state space realizations of the state and output feedback controllers as well as the dimension of the resulting controllers.

A. State Feedback

In the state feedback setting we consider the open-loop system

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right]$$

and would like to design a dynamic controller $\mathbf{u} = \mathbf{K}\mathbf{y}$. In this context, the system response matrices $\{\mathbf{R}, \mathbf{M}\}$ reduce to

$$\mathbf{R} = (zI - A - B_2\mathbf{K})^{-1}, \tag{8a}$$

$$\mathbf{M} = \mathbf{K}(zI - A - B_2\mathbf{K})^{-1}, \tag{8b}$$

and Theorem 1 reduces to the following corollary:

Corollary 1: For the state feedback system $\mathbf{P}$, the following are true:

(a) The affine subspace defined by

$$\begin{bmatrix} zI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I \tag{9a}$$

$$\mathbf{R}, \mathbf{M} \in \frac{1}{z}\mathcal{RH}_\infty \tag{9b}$$

parameterizes all system responses from $\boldsymbol{\delta_x}$ to $(\mathbf{x}, \mathbf{u})$, as defined in (8), achievable by an internally stabilizing state feedback controller $\mathbf{K}$.

(b) For any transfer matrices $\{\mathbf{R}, \mathbf{M}\}$ satisfying (9), the controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ is internally stabilizing and achieves the desired system response (8).

Our first result is to provide a state space realization of the state feedback controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ when the system response $\{\mathbf{R}, \mathbf{M}\}$ are FIR transfer matrices of horizon T, i.e., $\mathbf{R}, \mathbf{M} \in \mathcal{F}_T$.

Define by Z the *down shift* operator which is a square matrix with zeros everywhere apart from on the sub-diagonal which contains identity matrices of dimension $n \times n$. Next let $\mathcal{I}$ denote the matrix

$$\begin{bmatrix} I_n & 0_n & \dots & 0_n \end{bmatrix}^T$$

where there zero block is repeated $T - 1$ times.

Theorem 2: A minimal state space realization for the state feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y} = \mathbf{M}\mathbf{R}^{-1}\mathbf{x}$ with FIR horizon length T, is given by

$$\mathbf{K} = \left[\begin{array}{c|c} Z - \mathcal{I}\widehat{R} & -\mathcal{I} \\ \hline M_1\widehat{R} - \widehat{M} & M_1 \end{array}\right],$$

where

$$\widehat{R} = \begin{bmatrix} R_2, \dots, R_T \end{bmatrix} \quad \text{and} \quad \widehat{M} = \begin{bmatrix} M_2, \dots, M_T \end{bmatrix}$$

are the spectral elements of $\mathbf{R}$ and $\mathbf{M}$ respectively.

Proof: The full proof of the state feedback controller is ommitted as it follows as a special case of the output feedback controller presented in the next section. For completeness, we include here the realizations for $z\mathbf{M}$ and $z\mathbf{R}$:

The FIR filter $z\mathbf{M}$ with horizon T is described by

$$z\mathbf{M} = \sum_{k=1}^{T-1} \left(z^{-k} M_{k+1}\right) + M_1 + zM_0.$$

As M is strictly proper we have that $M_0 = 0$, and a state space realization for $z\mathbf{M}$ is

$$z\mathbf{M} = \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{M} & M_1 \end{array}\right].$$

As a consequence of constraint (9a), for $\mathbf{R}$ we have that $R_1 = I$, thus we have the realization

$$z\mathbf{R} = \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{R} & I \end{array}\right].$$

■

Note that the resulting controller has a state dimension that is linear in the horizon length of the FIR filter.

Corollary 2: Let the the state x in (1) have dimension n and the system response $\{\mathbf{R}, \mathbf{M}\}$ be FIR with horizon length T. Then the state dimension of $\mathbf{K}$ is $(T-1)n$.

B. Output Feedback

The output feedback controller takes the form $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$ with $\mathbf{M}, \mathbf{R}, \mathbf{N}$ strictly proper, and the transfer matrix $\mathbf{L}$ only required to be a proper. As in the state-feedback case, we assume that all transfer matrices of the system response are FIR with horizon T, i.e., $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L} \in \mathcal{F}_T$ The following state space realizations are used

$$\mathbf{L} = \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{L} & L_0 \end{array}\right], \quad \mathbf{M} = \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widetilde{M} & 0 \end{array}\right]$$

with $\widehat{L} = \begin{bmatrix} L_1, \dots, L_T \end{bmatrix}$, $\widetilde{M} = \begin{bmatrix} M_1, \dots, M_T \end{bmatrix}$ and

$$z\mathbf{N} = \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{N} & N_1 \end{array}\right], \quad \widehat{N} = \begin{bmatrix} N_2, \dots, N_T \end{bmatrix}.$$

Theorem 3: A minimal state space realization for the output feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y} = (\mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N})\mathbf{y}$ with horizon length T is given by

$$\mathbf{K} = \left[\begin{array}{cc|c} Z - \mathcal{I}\widehat{R} & \mathcal{I}\widehat{N} & \mathcal{I}N_1 \\ 0 & Z & \mathcal{I} \\ \hline \widetilde{M} & \widehat{L} & L_0 \end{array}\right].$$

Proof: Expand out $\mathbf{L} - \mathbf{M}(z\mathbf{R})^{-1}(z\mathbf{N})$ as

$$\left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{L} & L_0 \end{array}\right] - \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widetilde{M} & 0 \end{array}\right] \left[\begin{array}{c|c} Z - \mathcal{I}\widetilde{R} & -\mathcal{I} \\ \hline \widehat{R} & I \end{array}\right] \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{N} & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{L} & L_0 \end{array}\right] - \left[\begin{array}{cc|c} Z & \mathcal{I}\widehat{R} & \mathcal{I} \\ 0 & Z - \mathcal{I}\widehat{R} & -\mathcal{I} \\ \hline \widetilde{M} & 0 & 0 \end{array}\right] \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{N} & N_1 \end{array}\right],$$

apply the similarity transformation using

$$T = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$$

to the middle term and extract the minimal realization to get

$$\left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{L} & L_0 \end{array}\right] - \left[\begin{array}{c|c} Z - \mathcal{I}\widehat{R} & \mathcal{I} \\ \hline -\widetilde{M} & 0 \end{array}\right] \left[\begin{array}{c|c} Z & \mathcal{I} \\ \hline \widehat{N} & N_1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} Z & 0 & 0 & \mathcal{I} \\ 0 & Z - \mathcal{I}\widehat{R} & \mathcal{I}\widehat{N} & \mathcal{I}N_1 \\ 0 & 0 & Z & \mathcal{I} \\ \hline \widehat{L} & \widetilde{M} & 0 & L_0 \end{array}\right],$$

finally applying a second similarity transformation with

$$T = \begin{bmatrix} I & 0 & -I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

gives

$$\left[\begin{array}{ccc|c} Z & 0 & 0 & 0 \\ 0 & Z-\mathcal{I}\widehat{R} & \mathcal{I}\widehat{N} & \mathcal{I}N_1 \\ 0 & 0 & Z & \mathcal{I} \\ \hline \widehat{L} & \widetilde{M} & 0 & \widehat{L} \end{array}\right]$$

which leads to the minimal realization

$$\left[\begin{array}{cc|c} Z-\mathcal{I}\widehat{R} & \mathcal{I}\widehat{N} & \mathcal{I}N_1 \\ 0 & Z & \mathcal{I} \\ \hline \widetilde{M} & \widehat{L} & L_0 \end{array}\right].$$

■

Corollary 3: Let the the state x in (1) have dimension n and the system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ be FIR with horizon length T. Then the state dimension of the output feedback controller, $\mathbf{K}$, is $2(T-1)n$.

IV. $\mathcal{H}_2$ State Feedback

A. Controller Structure

Consider the dynamical system (1a) with $w[k] \equiv 0$ and assume it has been constructed from a network of m subsystems, where each sub system evolves according to

$$x_i[k{+}1] = A_{ii}x_i[k] + \sum_{j\in\mathcal{N}_i} A_{ij}x_j[k] + B_{ii}u_i[k], \;\; i = 1,\dots,m,$$

and all matrices and vectors are assumed to be of compatible dimensions. The system can then be compactly written as

$$x[k+1] = Ax[k] + Bu[k]$$

where x and u are the concatenated vectors containing the local states and control inputs. If there is no actuation at subsystem i then B_{ii} is a zero block. Note that in this simplified setting there is no coupling between the inputs hence B is a block diagonal matrix. In future work we will relax this assumption. We now consider the SLS formulation of the $\mathcal{H}_2$ state feedback problem:

$$\begin{aligned} \underset{\{\mathbf{R},\mathbf{M}\}}{\text{minimize}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} \quad & \text{(9a)}, \text{(9a)}, \{\mathbf{R}, \mathbf{M}\} \in \mathcal{S}_x \times \mathcal{S}_u \end{aligned} \tag{10}$$

where the sets $\mathcal{S}_x$ and $\mathcal{S}_u$ encode information sharing constraints on the closed loop system. Here we focus on the case where $\mathcal{S}_\star := \mathcal{L}_\star \cap \mathcal{F}_{T\star} \cap \mathcal{X}_\star$ with $\mathcal{L}$ defining a set of subspace (sparsity) constraints, $\mathcal{F}_T$ the FIR constraint, and $\mathcal{X}$ defining any other relevant convex set.[1] Define the standard performance weights $Q = C_1^T C_1$ and $R = D_{12}^T D_{12}$ and assume that $Q \succeq 0$, $R \succ 0$ and define the cost functional

$$\mathcal{J}^\dagger(M[k]) := \sum_{k=1}^{T-1} \text{Tr}\left(R[k]^T Q R[k] + M[k]^T R M[k]\right),$$

[1]We use the subscript $\star$ as shorthand to indicate both sets $\mathcal{S}_x$ and $\mathcal{S}_u$.

then (10) can be written in terms of its spectral components:

$$\begin{aligned} \underset{\{R[k],M[k]\}_{k=1}^T}{\text{min.}} \quad & \mathcal{J}^\dagger(R[k], M[k]) + \text{Tr}(R[T]^T Q_T R[T]) \\ \text{s.t.} \quad & R[k+1] = AR[k] + BU[k], \quad k = 1,\dots,T \\ & R[1] = I \\ & R[k] \in \mathcal{S}_x[k], \quad M[k] \in \mathcal{S}_u[k] \end{aligned} \tag{11}$$

Note that the set $\mathcal{S}_x[k]$ encodes constraints on the k^{th} spectral element and enforce locality constraints and the FIR constraint via $R[k+1] = 0$. In [14] *localized LQR*, LLQR, was introduced as a method for solving (10) and equivalently (11) in a decomposable manner. The idea is to take advantage of the fact that we have column separability in the objective function and the constraints. Let $x_j[k]$ denote the j^{th} column of $R[k]$ and $u_j[k]$ the j^{th} column of $M[k]$, then instead of solving the full problem (11) we can solve n problems of the form

$$\begin{aligned} \underset{\{x_j[k],u_j[k]\}_{k=1}^T}{\text{minimize}} \quad & \mathcal{J}_j(u_j[k]) + x_j^T[T] Q_T x_j[T] \\ \text{s.t.} \quad & x_j[k+1] = Ax_j[k] + Bu_j[k], \quad k = 1,\dots,T \\ & x_j[1] = e_j \\ & x_j[k] \in \mathcal{S}_x^j[k], \quad u_j[k] \in \mathcal{S}_u^j[k] \end{aligned}$$

for $j = 1,\dots,n$, where we define the decomposed cost functional

$$\mathcal{J}_j(u_j[k]) := \sum_{k=1}^{T-1} x_j[k]^T Q x_j[k] + u_j[k]^T R u_j[k], \tag{12}$$

e_j is the standard j^{th} basis vector for $\mathbb{R}^n$ and the superscript j on the sets $\mathcal{S}_\star$ denotes the j^{th} column of the appropriate sparsity constraint.

We now assume that the constraints $\mathcal{S}_\star$ enforce *d-localized* constraints (see [14]), i.e., that the support of the state and input constraints $\mathcal{S}_x$ and $\mathcal{S}_u$ satisfy

$$\text{supp}(\mathcal{S}_x) \subseteq \text{supp}(A)^{d-1}, \;\; \text{supp}(\mathcal{S}_u) \subseteq \text{supp}(A)^d. \tag{13}$$

Equation (13) leads to a natural notion of *boundary* states and control inputs, i.e., those states and control actions that when constrained to be zero ensure that the d-localized constraint as defined by the subspace $\mathcal{S}$ is satisfied. In particular, letting $\mathcal{A} := \text{supp}(A)$, we define the boundary state and control actions for the j-th subsystem as

$$\mathcal{B}_j^x := \{x_i \;| \text{supp}\left(\mathcal{A}^d - \mathcal{A}^{d-1}\right)_i \neq 0\}, \tag{14}$$

and

$$\mathcal{B}_j^u := \{u_i \;| \text{supp}\left(\mathcal{A}^{d+1} - \mathcal{A}^d\right)_i \neq 0\}, \tag{15}$$

respectively. To ease notation going forward, we simply write $x_j^{\mathcal{B}}$ and $u_j^{\mathcal{B}}$ to denote the boundary states and control actions for the the j-th subystem, i.e.

$$x_j^{\mathcal{B}} := \{x \mid x \in \mathcal{B}_j^x\}, \;\; u_j^{\mathcal{B}} := \{u \mid u \in \mathcal{B}_j^u\}.$$

This leads to the subproblems

$$\begin{aligned}
\underset{\{x_j[k],u_j[k]\}_{k=1}^T}{\text{minimize}} \quad & \mathcal{J}_j(u_j[k]) + x_j^T[T]Q_T x_j[T] && \text{(16a)}\\
\text{s.t.} \quad & x_j[k+1] = Ax_j[k] + Bu_j[k],\ k = 1,\dots,T \\
& x_j[1] = e_j \\
& x_j^{\mathcal{B}}[k] = 0, \quad k = 1,\dots,T && \text{(16b)}\\
& u_j^{\mathcal{B}}[k] = 0, \quad k = 1,\dots,T && \text{(16c)}
\end{aligned}$$

for $j = 1,\dots,n$. The goal now is to write the problem above in the form of a standard LQR problem. We will first take care of constraint (16c). Define the reduced vector $\tilde{u}_j[k] := u_j^{(1:\mathcal{B})}[k]$ where the superscript $(1:\mathcal{B})$ denotes the set of elements of u_j running from from the first to the boundary element and $\tilde{u}_j^{\mathcal{B}}$ is the last element of $\tilde{u}_j$. Similarly, let $\tilde{B}$ denote the matrix corresponding to the first $\mathcal{B}$ columns of B and $\tilde{R}$ the first $\mathcal{B}$ rows and columns of R. Let $\mathcal{J}_j'$ take the form of (12) but with the quadratic cost in u_j replaced with $\tilde{u}_j[k]^T\tilde{R}\tilde{u}_j[k]$. The LQR problem (16) now reduces to

$$\begin{aligned}
\underset{\{x_j[k],\tilde{u}_j[k]\}_{k=1}^T}{\text{minimize}} \quad & \mathcal{J}_j'(\tilde{u}_j[k]) + x_j[T]^T Q_T x_j[T] \\
\text{s.t.} \quad & x_j[k+1] = Ax_j[k] + \tilde{B}\tilde{u}_j[k],\ k = 1,\dots,T \\
& x_j[1] = e_j \\
& x_j^{\mathcal{B}}[k] = 0, \quad k = 1,\dots,T
\end{aligned}$$

which is almost in standard form.

Assumption 3: The boundary elements $x_j^{\mathcal{B}}$ can be directly controlled by the "actuators" on the boundary $\tilde{u}_j^{\mathcal{B}}$.

Next we remove the final non-standard constraint (16b). Consider the dynamics of the boundary states which are given by

$$x_j^{\mathcal{B}}[k+1] = A^{\mathcal{B}'}x_j[k] + \tilde{B}^{\mathcal{B}'}\tilde{u}_j^{\mathcal{B}}[k].$$

Here, $A^{\mathcal{B}'}$ refers to the row in A corresponding to boundary state. Under the premise of assumption 3 the boundary control can be chosen to cancel out the state dynamics by selecting

$$\tilde{u}_j^{\mathcal{B}} = -A^{\mathcal{B}'}x_j[k]\frac{1}{\tilde{B}^{\mathcal{B}'}} \tag{17}$$

which by definition satisfies (16b). Finally we rewrite the cost function to take into account the fact that the boundary control law (17) can be expressed in terms of the state. The quadratic control penalty can be decomposed into

$$\tilde{u}_j^T\tilde{R}\tilde{u}_j = \hat{u}_j^T\mathcal{R}\hat{u}_j^T + (u_1^{\mathcal{B}})^T R^{\mathcal{B}} u_1^{\mathcal{B}},$$

where the second quadratic term can be written in terms of x_j by substituting in the control action (17). The resulting LQR problem is then

$$\begin{aligned}
\underset{\{\hat{x}_j[k],\hat{u}_j[k]\}_{k=1}^T}{\text{minimize}} \quad & \mathcal{J}_j(\hat{u}_j[k]) + \hat{x}_j[T]^T\mathcal{Q}_T\hat{x}_j[k] \\
\text{s.t.} \quad & \hat{x}_j[k+1] = \hat{A}\hat{x}_j[k] + \tilde{B}\hat{u}_j[k],\ k = 1,\dots,T \\
& \hat{x}_j[1] = e_j && \text{(18)}
\end{aligned}$$

with the cost function defined as

$$\mathcal{J}_j(\hat{u}_j[k]) = \sum_{k=1}^{T-1}\hat{x}_j[k]^T\mathcal{Q}\hat{x}_j[k] + \hat{u}[k]^T\mathcal{R}\hat{u}[k],$$

where

$$\mathcal{Q} := (Q + \Gamma^T(A^{\mathcal{B}'})^T R A^{\mathcal{B}'}\Gamma), \quad \Gamma := \left(\tilde{B}^{\mathcal{B}'}\right)^{-1},$$

and $\hat{x}_j[k] := x_j^{(1:\mathcal{B})}[k]$. Note now that the n problems defined by (18) are in standard LQR form.

B. Solution Complexity

The dynamic programming solution to the finite-time LQR problem gives an optimal control law defined by $u^\star[k] = K_k x[k]$ where the control gain at each time step is computed from the following procedure [19, §4.1]:

1) Let $P_T := Q_T$.
2) For $k = T,\dots,1$ solve the Riccati recursion
$$P_{k-1} = Q + A^T P_k A - A^T P_k B(R + B^T P_k B)^{-1}B^T P_k A.$$
3) For $k = 0,\dots,T-1$ construct the controller gain
$$K_k := -(R + B^T P_{k+1}B)^{-1}B^T P_{k+1}A.$$

The complexity of the solution is $\mathcal{O}(Tn^3)$, this intuitively makes sense as it requires solving T Riccati equations corresponding to an n-state system. Define N_j to be the dimension of $\hat{x}_j$ for $j = 1,\dots,n$, then the complexity of constructing the $\mathcal{H}_2$-state feedback controller via a dynamic programming is $\mathcal{O}(nTN_j^3)$, however as the problem decomposes perfectly, in practice one solves n dynamic programming problems (in parallel) of $\mathcal{O}(TN_j^3)$ and in practice $N_j \ll n$.

For the infinite horizon LQR problem ($T = \infty$), the optimal control strategy is not time-dependant and is given by control policy

$$u^\star[k] = -(R + B^T PB)^{-1}B^T PAx[k].$$

Translating this to the reduced order, decomposed, SLS problem (18), it is easily seen that the complexity of the controller synthesis problem requires solving n Ricatti equations (potentially in parallel) with a cost of $\mathcal{O}(N_j^3)$ for $j = 1,\dots,n$.

V. Conclusions

Explicit state space realizations and dimensionality scalings for state and output feedback controllers in the system level synthesis framework have been derived. For the $\mathcal{H}_2$ state feedback case we showed that the controller could be constructed by solving a series of reduced order LQR problems. Future work will involve looking at the LQG problem as well as removing the assumption of no input coupling.

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