A Heuristic for Sub-Optimal $\mathcal{H}_2$ Decentralized Control subject to Delay in Non-Quadratically-Invariant Systems

Nikolai Matni and John C. Doyle

***Abstract*— Inspired by potential applications to the smart grid, we develop a heuristic for sub-optimal, but acceptable, control of decentralized systems subject to *non-quadratically invariant* (non-QI) delay patterns. We do so by exploiting a recently developed solution to the decentralized $\mathcal{H}_2$ model matching problem subject to delays, which decomposes the controller into a centralized, but delayed, component and a decentralized FIR component. In particular, we present an iterative procedure that exploits this decomposition to design a sub-optimal decentralized $\mathcal{H}_2$ controller for non-QI systems that is guaranteed *a priori* to be stable, and to perform no worse than a controller computed with respect to a QI subset of the non-QI constraint set. We then apply this procedure to a smart-grid frequency regulation problem.**

I. INTRODUCTION

Decentralized control problems arise when several decision makers, or controllers, need to determine their actions, or inputs, based only on a subset of the total information available about the system. These types of problems arise in areas as diverse as physiology, economics and the power grid. A particular class of decentralized control problems that has received a significant amount of attention over the past few decades is that of optimal $\mathcal{H}_2$ (or LQG) control subject to delay constraints. In this case, the information constraints can be interpreted as arising from a communication graph, in which edge weights between nodes correspond to the delay required to transmit information between them.

For the special case of the one-step delay information sharing pattern, the $\mathcal{H}_2$ problem was solved in the 1970s using dynamic programming [1], [2], [3]. For more complex delay patterns, the separation principle fails [4], [5], [6], making extensions beyond the state feedback case [7], [8] difficult, although recent work [9] provides two dynamic programming decompositions for the general delayed sharing model.

In [10] it was shown that quadratically invariant (QI) constraint sets are necessary and sufficient for a decentralized optimal control problem to be amenable to convex optimization. In [11], these results were specialized to the communication delay case, and it was shown that a sufficient (and under mild assumptions, necessary) condition for a constraint set to be QI is that the controllers are able to exchange information at least as fast as the dynamics propagate through the plant.

In cases where this condition is met, the output feedback $\mathcal{H}_2$ problem with communication delays has been solved using vectorization [10] or linear matrix inequalities (LMI) [12], [13] techniques, and most recently using an extension of spectral factorization [14]. Additionally, methods and/or solutions exist for special instances of this problem, such as the two-player systems considered in [15], [16], and spatially invariant systems [17], [18], [19].

In most mechanical systems, the QI condition is easily met, so long as appropriately reliable communication networks can be established. In the case of the power-grid, however, both the communication and physical systems are driven by electromagnetic energy and could propagate on similar time scales.

Furthermore, a key characteristic of next-generation power distribution systems is the incorporation of distributed renewable energy sources (DRES), such as photovoltaics, wind turbines and electric vehicles. DRESs differ from their traditional counterparts in two significant ways: (i) they are not frequency coupled to the grid and (ii) their generation is intermittent and unreliable. With thousands, and even millions, of these DRESs plugging in to and out of the grid, the possibility of the entire system being destabilized is very real. In other words, incorporating these generators into the grid eliminates the system's inherent stability. Therefore, in the new smart grid, sophisticated low-level control will not only be beneficial to performance, but necessary for safety and stability[20].

Inspired by this potential application, we look to develop a heuristic for sub-optimal, but acceptable, control of decentralized systems subject to *non-QI* delay patterns. As far as the authors are aware, no other works in this spirit exist in the literature. In [14], the QI problem is solved by decomposing the controller into a centralized, but delayed, component and a decentralized FIR component. Our main contribution is in developing an iterative procedure that exploits this decomposition to design a sub-optimal decentralized $\mathcal{H}_2$ controller for non-QI systems that is guaranteed *a priori* to be stable, and to perform no worse than a controller computed with respect to a QI subset of the non-QI constraint set. We then apply this procedure to a smart-grid frequency regulation problem.

This paper is organized as follows: Section II presents the general problem to be studied, and presents the solution from [14] for the QI case. Section III presents our heuristic for the non-QI case, and Section IV applies this heuristic to a smart-grid frequency regulation problem that is subject to non-QI communication constraints. Section V ends with conclusions, and suggestions for future work.

N. Matni and J.C. Doyle are with the Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, CA. nmatni@caltech.edu.

II. Preliminaries

1) $\mathcal{H}_2$ Preliminaries: Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of complex numbers. A function $G : (\mathbb{C} \bigcup \{\infty\}) \backslash \mathbb{D} \to \mathbb{C}^{p \times q}$ is in $\mathcal{H}_2$ if it can be expanded as

$$G(z) = \sum_{i=0}^{\infty} \frac{1}{z^i} G_i$$

where $G_i \in \mathbb{C}^{p \times q}$ and $\sum_{i=0}^{\infty} \mathrm{Tr}(G_i G_i^*) < \infty$. Define the conjugate of G by

$$G(z)^{\sim} = \sum_{i=0}^{\infty} z^i G_i^*$$

$\mathcal{H}_2$ is a Hilbert space with inner product given by

$$\begin{aligned} < G, H > &= \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Tr}(G(e^{j\theta}) H(e^{j\theta})^{\sim}) d\theta \\ &= \sum_{i=0}^{\infty} \mathrm{Tr}(G_i H_i^*), \end{aligned}$$

where the last equality follows from Parseval's identity.

Finally, if $\mathcal{M}$ is a subspace of $\mathcal{H}_2$, denote the orthogonal projection onto $\mathcal{M}$ by $\mathbb{P}_{\mathcal{M}}$.

2) Decentralized Model Matching: Quadratically Invariant Case: Let P be a stable discrete-time plant given by

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad (1)$$

with inputs of dimension p_1, p_2 and outputs of dimension q_1, q_2. We restrict attention to stable plants for simplicity. These methods could also be applied to an unstable plant if a stable stabilizing nominal controller can be found, as in [10]. We note that this task may be non-trivial, with strong guarantees existing only in the sparsity constrained setting [21].

To ensure the existence of stabilizing solutions to the appropriate Riccati equations (note that stabilizability and detectability of (A, B_2, C_2) is implied by the assumption of a stable plant), we assume

- $D_{12}^T D_{12} > 0$,
- $D_{21} D_{21}^T > 0$,
- $C_1^T D_{12} = 0$
- $B_1 D_{21}^T = 0$

For $N \geq 1$, define the space of strictly proper finite impulse response (FIR) transfer matrices by $\mathcal{X}_N = \oplus_{i=1}^{N} \frac{1}{z^i} \mathbb{C}^{p_2 \times q_2}$. Note that in the following, we sometimes suppress the subscript and write $\mathcal{X}_N = \mathcal{X}$ when N is clear from context. We can therefore decompose $\frac{1}{z}\mathcal{H}_2$ into orthogonal subspaces as

$$\frac{1}{z}\mathcal{H}_2 = \mathcal{X}_N \oplus \frac{1}{z^{N+1}} \mathcal{H}_2,$$

In this paper, we are concerned with controller constraints described by delay patterns that are imposed by *strongly connected communication graphs*. As such, let $\mathcal{R}_p$ be the space of proper real rational transfer matrices, and $\mathcal{S} \subset \frac{1}{z}\mathcal{R}_p$ be a subspace of the form

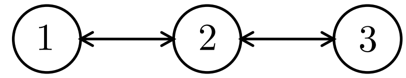

Fig. 1: The graph depicts the communication structure of the three-player chain problem. Edge weights (not shown) indicate the delay required to transmit information between nodes.

$$\mathcal{S} = \mathcal{Y} \oplus \frac{1}{z^{N+1}} \mathcal{R}_p \quad (2)$$

where $\mathcal{Y} = \oplus_{i=1}^{N} \frac{1}{z^i} \mathcal{Y}_i \subset \oplus_{i=1}^{N} \frac{1}{z^i} \mathbb{R}^{p_2 \times q_2} \subset \mathcal{X}_N$. Specifically, this implies that every decision-making agent has access to *all* measurements that are at least $N+1$ time-steps old.

We can therefore partition the measured outputs y and control inputs u according to the dimension of the subsystems:

$$y = [\; y_1^T \quad \cdots \quad y_m^T]^T \quad u = [\; u_1^T \quad \cdots \quad u_n^T]^T$$

and then further partition each constraint set $\mathcal{Y}_i$ as

$$\mathcal{Y}_i = \begin{bmatrix} \mathcal{Y}_i^{11} & \cdots & \mathcal{Y}_i^{1m} \\ \vdots & \ddots & \vdots \\ \mathcal{Y}_i^{n1} & \cdots & \mathcal{Y}_i^{nm} \end{bmatrix}, \quad (3)$$

where

$$\mathcal{Y}_i^{jk} = \begin{cases} \mathbb{R}^{p_2^j \times q_2^k} & \text{if } u_j \text{ has access to } y_k \text{ at time } i \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

and $\sum_{j=1}^{n} p_2^j = p_2$, $\sum_{k=1}^{m} q_2^k = m$.

Example 1: Consider the three player chain problem as illustrated in Figure 1, with communication delay τ_c between nodes. Then

$$\begin{aligned} \mathcal{S} &= \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^{1+\tau_c}}\mathcal{R}_p & \frac{1}{z^{1+2\tau_c}}\mathcal{R}_p \\ \frac{1}{z^{1+\tau_c}}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p & \frac{1}{z^{1+\tau_c}}\mathcal{R}_p \\ \frac{1}{z^{1+2\tau_c}}\mathcal{R}_p & \frac{1}{z^{1+\tau_c}}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix} \\ &= \oplus_{i=1}^{2\tau_c} \frac{1}{z^i}\mathcal{Y}_i \oplus \frac{1}{z^{2\tau_c+1}}\mathcal{R}_p \end{aligned} \quad (5)$$

with

$$\mathcal{Y}_i = \begin{cases} \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} & \text{for } i \leq \tau_c \\ \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} & \text{for } \tau_c < i \leq 2\tau_c \end{cases},$$

where, for compactness, * is used to denote a space of appropriately sized real matrices. In this setting, every decision maker then has access to all measurements that are at least $2\tau_c + 1$ time-steps old.

The decentralized control problem of interest is to design a controller $K \in \mathcal{S}$ so as to minimize the closed loop $\mathcal{H}_2$ norm of the system:

$$\begin{aligned} \underset{K}{\text{minimize}} \quad & \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|_{\mathcal{H}_2} \\ & \text{s.t. } K \in \mathcal{S} \end{aligned} \quad (6)$$

In [10], it was shown that a necessary and sufficient condition to be able to pass to the Youla parameter $Q =$

$K(I-P_{22}K)^{-1}$ in (6) is that the constraint set be *quadratically invariant*.

Definition 1: A set $\mathcal{S}$ is *quadratically invariant* under P_{22} if

$$KP_{22}K \in \mathcal{S} \text{ for all } K \in \mathcal{S}$$

Under this assumption on $\mathcal{S}$ and P_{22}, we can pass without loss to the Youla domain. Since K is strictly proper and stabilizing, Q must be strictly proper and stable; thus (6) can be reduced to the following model matching problem:

$$\begin{array}{rl} \text{minimize}_Q & \|P_{11}+P_{12}QP_{21}\|_{\mathcal{H}_2} \\ & \text{s.t.}\, Q \in \mathcal{S} \bigcap \frac{1}{z}\mathcal{H}_2 \end{array} \tag{7}$$

For technical simplicity, all controllers in this paper are assumed to be strictly proper – the results extend to non-strictly proper controllers, but the resulting formulas are more complicated. Although this problem admits several solutions [9], [10], [12], [13], we follow the one presented in [14].

Let X, Y be the stabilizing solutions to the following Riccati Equations

$$\begin{aligned} X &= C_1^TC_1 + A^TXA - (A^TXB_2 + C_1^TD_{12}) \times \\ &\quad \Omega^{-1}(A^TXB_2 + C_1^TD_{12})^T \\ Y &= B_1^TB_1 + AYA^T - (AYC_2^T + B_1D_{21}^T) \times \\ &\quad \Psi^{-1}(AYC_2^T + B_1D_{21}^T)^T \end{aligned}$$

where $\Omega := D_{12}^TD_{12} + B_2^TXB_2$, and $\Psi := D_{21}D_{21}^T + C_2YC_2^T$. Define the regulator and filter gains, respectively, as

$$\begin{aligned} K &= -\Omega^{-1}(B_2^TXA + D_{12}^TC_1) \\ L &= -(AYC_2^T + B_1D_{21}^T)\Psi^{-1} \end{aligned}$$

and the auxiliary matrix T by

$$T = \Omega^{1/2}\left[\begin{array}{c|c} A & L \\ \hline K & 0 \end{array}\right]\Psi^{1/2}. \tag{8}$$

Finally, let W_L and W_R be left and right spectral factors for $P_{12}^{\sim}P_{12}$ and $P_{21}P_{21}^{\sim}$ such that

$$P_{12}^{\sim}P_{12} = W_L^{-\sim}W_L^{-1}, \quad P_{21}P_{21}^{\sim} = W_R^{-1}W_R^{-\sim}$$

We first present the classical solution to the delayed model matching problem, from which the decentralized solution is then constructed.

Theorem 1: The optimal solution to the delayed model matching problem

$$\begin{array}{rl} \text{minimize}_Q & \|P_{11}+P_{12}QP_{21}\|_{\mathcal{H}_2} \\ & \text{s.t.}\, Q \in \frac{1}{z^{N+1}}\mathcal{H}_2 \end{array} \tag{9}$$

is given by

$$Q_N = -W_L\mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2}(T)W_R$$

Theorem 2: (From [14]) The optimal solution to (7) is given by

$$Q^* = U^* + V^*$$

where $V^* \in \mathcal{Y}$ is the unique minimizer of

$$\|G(V)\|_{\mathcal{H}_2}^2 + 2 < G(V), T > \tag{10}$$

with $G(V) = \mathbb{P}_{\mathcal{X}}(W_L^{-1}VW_R^{-1})$, and

$$U^* = Q_N - W_L\mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2}(W_L^{-1}V^*W_R^{-1})W_R \in \frac{1}{z^{N+1}}\mathcal{H}_2. \tag{11}$$

The optimal cost is then given by

$$\begin{aligned} J(Q^*,P) &= \|P_{11}+P_{12}Q_NP_{21}\|_{\mathcal{H}_2}^2 \\ &\quad + \|G(V^*)\|_{\mathcal{H}_2}^2 + 2 < G(V^*), T > \end{aligned} \tag{12}$$

The assumption of a strongly connected graph is key in the above, as it allows for the optimal controller Q^* to be decomposed as the direct sum of a FIR filter V^*, and a delayed, but centralized, component U^* that depends only on *globally available* information.

The FIR component V^* of the optimal decentralized controller can be computed by solving a quadratic program. For ease of notation, let $G_i(V) = G_i$, and $H = W_L^{-1}$, $J = W_R^{-1}$. Note that H and J can be expanded as $H = \sum_{i=0}^{\infty}\frac{1}{z^i}H_i$ and $J = \sum_{i=0}^{\infty}\frac{1}{z^i}J_i$. Similarly, T and V admit the expansions $T = \sum_{i=1}^{\infty}\frac{1}{z^i}T_i$, and $V = \sum_{i=1}^{N}\frac{1}{z^i}V_i \in \mathcal{Y}$, with each $V_i \in \mathcal{Y}_i$.[1]

Lemma 1: The FIR transfer matrix $G(V)$ can be written

$$G(V) = \sum_{i=1}^{N}\frac{1}{z^i}G_i, \text{ with } G_i = \sum_{\substack{j,l\geq 0,k\geq 1 \\ j+k+l=i}} H_jV_kJ_l$$

and, applying Parseval's identity to (10), we can formulate the optimization problem as

$$\begin{array}{rl} \text{minimize}_{\mathrm{V}} & \sum_{i=1}^N \mathrm{Tr}G_iG_i^* \quad +2\sum_{i=1}^N \mathrm{Tr}G_iT_i^* \\ \text{s.t.} & V_i \in \mathcal{Y}_i \end{array} \tag{13}$$

Remark 1: It was shown in [14] that (13) is a convex quadratic program with a unique solution.

III. Decentralized Model Matching: Non-Quadratically-Invariant Case

In cases where the constraint set is *not* QI, the optimization problem (6) becomes much more difficult as we cannot directly pass to the Youla domain. In fact, in this case, it is not guaranteed that the optimal control law is even linear.

In this section, we present a heuristic for decentralized sub-optimal control that appears to work well in practice. We begin by introducing the notion of a QI cover for a non-QI constraint set:

Definition 2: A *QI cover* $\bar{\mathcal{S}}_{QI}$ of a non-QI constraint set $\mathcal{S}$ is a set of subsets $\{\mathcal{S}_i\} \subset \mathcal{S}$ such that

1) $\mathcal{S}_i$ is QI under P_{22}
2) $\bigcup_{\mathcal{S}_i\in\bar{\mathcal{S}}_{QI}} \mathcal{S}_i = \mathcal{S}$

Furthermore, for the types of constraint sets $\mathcal{S}$ that we consider in this paper (induced by strongly connected communication graphs), a QI cover always exists:

Proposition 1: Let $\mathcal{S}$ be a constraint set of the form (2). Then there exists a QI cover $\bar{\mathcal{S}}_{QI}$ for $\mathcal{S}$.

Proof: If $\mathcal{S}$ is QI under P_{22}, simply set $\bar{\mathcal{S}}_{QI} = \mathcal{S}$. Otherwise, let $E_i = [0,...,I,...,0]^T$, where the identity matrix is in the

[1] The component matrices H_i, J_i and T_i can be easily computed via state space methods, c.f. [14]

i^{th} position, and taken to be of appropriate dimension based on context. Recall that $\mathcal{S}$ can be expanded as

$$\mathcal{S} = \oplus_{i=1}^{N} \frac{1}{z^i}\mathcal{Y}_i \oplus \frac{1}{z^{N+1}}\mathcal{R}_p \tag{14}$$

where each $\mathcal{Y}_i \subset \mathbb{R}^{p_2 \times q_2}$ is partitioned as in (3). Let $i_{ab} := \min\{i \in \{1,...,N+1\} \,|\, \mathcal{Y}_i^{ab} \neq 0\}$. Let

$$\mathcal{S}^{ab} := \oplus_{i=i_{ab}}^{N} \frac{1}{z^i} E_a \mathcal{Y}_i^{ab} E_b^T \oplus \frac{1}{z^{N+1}}\mathcal{R}_p,$$

and

$$\bar{\mathcal{S}}_{QI} := \{\mathcal{S}^{ab} \,|\, i_{ab} \leq N\}.$$

The claim is that $\bar{\mathcal{S}}_{QI}$ is a QI-cover of $\mathcal{S}$. Clearly $\sum_{\mathcal{S}_i \in \bar{\mathcal{S}}_{QI}} \mathcal{S}^{ab} = \mathcal{S}$, and so it suffices to show that each $\mathcal{S}^{ab}$ is QI under P_{22}, or equivalently that $KP_{22}K \in \mathcal{S}^{ab}$ for all $K \in \mathcal{S}^{ab}$. For ease of notation, we will write $P_{22} = G$. Let $K \in \mathcal{S}^{ab}$ be arbitrary. Since both G, $K \in \frac{1}{z}\mathcal{H}_2$ they admit the expansions

$$G = \sum_{i=1}^{\infty} \frac{1}{z^i} G_i, \;\; K = \sum_{i=1}^{\infty} \frac{1}{z^i} K_i$$

for appropriately chosen matrices G_i, K_i. Similary, we can expand KGK as

$$KGK = \sum_{i=1}^{\infty} \frac{1}{z^i} (KGK)_i$$

where

$$(KGK)_i = \sum_{\substack{j,k,l \geq 1 \\ j+k+l=i}} K_j G_k K_l.$$

In order to show $KGK \in \mathcal{S}^{ab}$, it suffices to show that $(KGK)_i \in E_a \mathcal{Y}_i^{ab} E_b^T$ for $i = 1,...,N$. Given that $K \in \mathcal{S}^{ab}$, for $i = 1,...,N$ we can write

$$K_i = \begin{cases} E_a \tilde{K}_i E_b^T & \text{if } i \geq i_{ab} \\ 0 & \text{otherwise} \end{cases},$$

for appropriately chosen $\tilde{K}_i \in \mathcal{Y}_i^{ab}$. Therefore for $j,k,l \geq 1$, $j+k+l = i$, $i = 1,...,N$ we have

$$\begin{aligned} K_j G_k K_l &= E_a \tilde{K}_j E_b^T G_k E_a \tilde{K}_l E_b^T \\ &= E_a \underbrace{\tilde{K}_j G_k^{ba} \tilde{K}_l}_{\in \mathcal{Y}_i^{ab}} E_b^T \in E_a \mathcal{Y}_i^{ab} E_b^T. \end{aligned}$$

As each term in the expansion of $(KGK)_i$ is in $E_a \mathcal{Y}_i^{ab} E_b^T$, it follows that $(KGK)_i \in E_a \mathcal{Y}_i^{ab} E_b^T$, proving the claim. ■

Remark 2: An intuitive interpretation of QI in the delay constrained case is that the delay pattern between controllers needs to be such that it removes any incentive to signal through the plant. With this in mind, we can easily understand why $\mathcal{S}^{ab} \in \bar{\mathcal{S}}_{QI}$ in the proof above is QI: during the initial FIR window (i.e. for $i = 1,...,N$) only one controller is active, and thus has no need to communicate (be it through the plant or otherwise) with the other controllers.

Remark 3: This construction is by no means unique, and "better" (for our heuristic) constructions will usually exist. This point will be illustrated in the examples, and algorithms for constructing better QI covers (possibly exploiting the methods developed in [22]) are the subject of future work.

Example 2: Consider the previously described three player chain problem with $\mathcal{S}$ as in (5) , with $\tau_c = 3$, and let

$$P_{22} \in \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^2}\mathcal{R}_p & \frac{1}{z^3}\mathcal{R}_p \\ \frac{1}{z^2}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p & \frac{1}{z^2}\mathcal{R}_p \\ \frac{1}{z^3}\mathcal{R}_p & \frac{1}{z^2}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix}.$$

It is easily verified that $\mathcal{S}$ is *not* QI under P_{22}. A QI cover for $\mathcal{S}$ can be constructed as described in the proof of the previous lemma, with cardinality 7. However, another QI cover of smaller cardinality, and with less restrictive subsets $\mathcal{S}_i$, is given by

$$\bar{\mathcal{S}}_{QI} = \left\{ \begin{bmatrix} \frac{1}{z^3}\mathcal{R}_p & \frac{1}{z^4}\mathcal{R}_p & \frac{1}{z^7}\mathcal{R}_p \\ \frac{1}{z^4}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p & \frac{1}{z^4}\mathcal{R}_p \\ \frac{1}{z^7}\mathcal{R}_p & \frac{1}{z^4}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix}, \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^4}\mathcal{R}_p & \frac{1}{z^7}\mathcal{R}_p \\ \frac{1}{z^4}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p & \frac{1}{z^4}\mathcal{R}_p \\ \frac{1}{z^7}\mathcal{R}_p & \frac{1}{z^5}\mathcal{R}_p & \frac{1}{z^3}\mathcal{R}_p \end{bmatrix} \right\}$$

We are now in a position to present our algorithm for sub-optimal control subject to non QI constraints:

Algorithm 1: Consider a plant P as in (1) and a non-QI constraint set $\mathcal{S}$ of the form (2). Then the algorithm proceeds as follows:

1) Given a non-QI constraint set $\mathcal{S}$ and plant P, compute a QI cover $\bar{\mathcal{S}}_{QI}$.
2) For each $\mathcal{S}_i \in \bar{\mathcal{S}}_{QI}$, compute Q_i according to Theorem 2 and Lemma 1 to obtain costs $J(Q_i, P)$.
3) Select $i^* = \arg\min_i J(Q_i, P)$ and set $K = Q_{i^*}(I + P_{22}Q_{i^*})^{-1}$. Apply controller K to plant P to obtain the closed loop plant $\bar{P}$.
4) Set $P = \bar{P}$ and $\bar{\mathcal{S}}_{QI} = \bar{\mathcal{S}}_{QI} \backslash \mathcal{S}_{i^*}$. If $\bar{\mathcal{S}}_{QI} \neq \emptyset$, return to 2.

Remark 4: Since this method is a greedy algorithm, it is guaranteed to perform at least as well as optimizing with respect to the best QI subset $\mathcal{S}_i^* \in \bar{\mathcal{S}}_{QI}$. After the first iteration, the closed loop norm of the system is precisely that obtained when optimizing with respect to $\mathcal{S}_i^*$– at each subsequent iteration, $Q_i = 0$ is always a feasible controller, and hence the sequence of costs $J(Q_{i^*}, P)$ is non-increasing.

To explain the intuition behind this heuristic, assume that $\bar{\mathcal{S}}_{QI}$ is generated as in the proof of Proposition 1. At each step, the algorithm introduces artificial inter-controller delay by optimizing with respect to a QI constraint set $\mathcal{S}_i \in \bar{\mathcal{S}}_{QI}$: this allows for easy computation of a decentralized controller (that is optimal within $\mathcal{S}_i$) at the expense of a higher cost. Then, as the algorithm iterates, each such optimization computes the optimal FIR component of a local controller *assuming all other controllers are inactive* during this initial timeframe, and adjusts the rest of the controller according to (11).

In practice, we observe that this heuristic works extremely well when the dynamic coupling between the plants is weak during the FIR window $i = 1,...,N$, but does not perform as well otherwise. We illustrate these ideas with the following toy example

Example 3: Let

$$P^1 = \left[\begin{array}{cc|c:cc} .1 & .8 & \multirow{2}{*}{$I_{2\times 2}$} & 1 & -1 \\ -.8 & .1 & & -1 & 1 \\ \hline 1 & -1 & \multirow{2}{*}{$0_{2\times 2}$} & \multicolumn{2}{c}{\multirow{2}{*}{$I_{2\times 2}$}} \\ -1 & 1 & & & \\ \hdashline \multicolumn{2}{c|}{I_{2\times 2}} & I_{2\times 2} & \multicolumn{2}{c}{0_{2\times 2}} \end{array}\right]$$

$$\implies P_{22}^1 \in \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \\ \frac{1}{z}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix},$$

$$P^2 = \left[\begin{array}{cc|c:c} .9 & 0 & \multirow{2}{*}{$I_{2\times 2}$} & \multirow{2}{*}{$10I_{2\times 2}$} \\ 0 & .9 & & \\ \hline 1 & -1 & \multirow{2}{*}{$0_{2\times 2}$} & \multirow{2}{*}{$I_{2\times 2}$} \\ -1 & 1 & & \\ \hdashline \multicolumn{2}{c|}{I_{2\times 2}} & I_{2\times 2} & 0_{2\times 2} \end{array}\right],$$

$$\implies P_{22}^2 \in \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^2}\mathcal{R}_p \\ \frac{1}{z^2}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix},$$

and

$$S = \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^5}\mathcal{R}_p \\ \frac{1}{z^5}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix}.$$

Then $\mathcal{S}$ is not QI under either P_{22}^i. QI covers under P_{22}^1 and P_{22}^2 are given by, respectively,

$$\bar{\mathcal{S}}^1 = \left\{ \begin{bmatrix} \frac{1}{z^3}\mathcal{R}_p & \frac{1}{z^5}\mathcal{R}_p \\ \frac{1}{z^5}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix}, \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^5}\mathcal{R}_p \\ \frac{1}{z^5}\mathcal{R}_p & \frac{1}{z^3}\mathcal{R}_p \end{bmatrix} \right\}$$

$$\bar{\mathcal{S}}^2 = \left\{ \begin{bmatrix} \frac{1}{z^2}\mathcal{R}_p & \frac{1}{z^5}\mathcal{R}_p \\ \frac{1}{z^5}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix}, \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^5}\mathcal{R}_p \\ \frac{1}{z^5}\mathcal{R}_p & \frac{1}{z^2}\mathcal{R}_p \end{bmatrix} \right\}.$$

Note that although in this example, the matrices B_1, D_{21}, C_1 and D_{12} do not satisfy our assumptions from Section II.2, stabilizing equations to the necessary Riccati equations do exist, and thus our methods are still applicable.

Applying our heuristic to P_{22}^2 with $\bar{\mathcal{S}}^2$, we obtain a cost of 2.791, which is the same cost that is achieved if we optimize with respect to the non QI constraint set $\mathcal{S}$ (i.e. by applying Theorem 2 and Lemma 1 with $Q \in \mathcal{S}$). In this case we note that there is no coupling between the two plants during the first time step, which is precisely the component of the FIR filter that we are iterating over.

Conversely, if we apply our heuristic to P_{22}^1 with $\bar{\mathcal{S}}^1$, we obtain a cost of 23.668, which is higher than 23.211, the cost achieved when optimizing with respect to the non QI constraint set $\mathcal{S}$. In this case, we see that there is coupling between the two plants from the first time step on, and we are iterating over the first two components of the FIR filter: this leads to a loss in performance when optimizing the local FIR components individually.

Remark 5: A glaring issue with this heuristic is that of scalability. If $|\bar{\mathcal{S}}| \gg 1$ then a large number of loops need to be closed, resulting in an extremely high order controller – this issue will be the subject of future work.

IV. Smart-Grid Application

It was shown in [11] that a sufficient condition for a constraint set induced by delay patterns to be QI is that the propagation delay between plants be at least as long as the communication delay between controllers. In many physical systems, this requirement can easily be achieved – one system where it is not obvious if this will hold true is the power grid. As was explained in the introduction, if renewables are to be safely incorporated into the new smart-grid, low-level decentralized control will be necessary, even if the system is not QI.

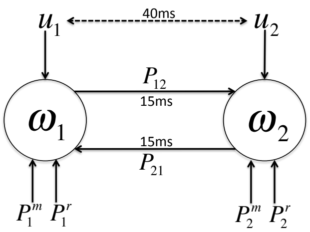

Fig. 2: The smart-grid frequency regulation problem considered in Section IV: the goal is to regulate frequency deviations ω_i, despite both mechanical (P_i^m) and renewables (P_i^r) based disturbances, using a decentralized controller $u = [u_1, u_2]^T$ subject to non-QI communication constraints.

As far as the authors are aware, the relevant time-scales have not been identified yet in the power-systems literature, and so we use educated guesses for the value of the delays. We consider a two bus power network model (see Figure 2), modified from [23], where the goal is to suppress the frequency deviations ω_i, despite mechanical and renewables based disturbances, by varying the response of frequency-insensitive but controllable loads (with control input u_i).

First, we introduce some notation and parameter values: (i) Δt=.005s: sampling period, (ii) $M_1 = 8.1564$s, $M_2 = 6.000$s: generator inertia constant (s), (iii) $D_1 = 4.0782$pu, $D_2 = 3.000$pu: frequency sensitive load damping constant, (iv) $\omega^0 = 120\pi \text{ rad}^{-1}$: common nominal frequency, (v) $B_{12} = B_{21}$=16.850pu: branch flow constant, (vi) P_i^r (W) renewable power disturbances, (vii) $\gamma_i^r = \frac{M_i}{\Delta t}$: renewable power disturbance scaling constant, (viii) P_i^m (W) mechanical power disturbances, (ix) $\gamma_i^m = \frac{2M_i}{\Delta t}$: mechanical power disturbance scaling constant, (x) ν_i (rad^{-1}): sensor noise, and finally (xi) $\epsilon = \frac{.05}{\Delta t}$(pu/s): transmission line dissipation constant.

The dynamics of the system are then given by (15), where the state vector x, disturbance vector ν, and control vector u are given by

$$\begin{aligned} x &= [\ \omega_1 \quad \omega_2 \quad P_{12} \quad P_{21}\ ]^T \\ \nu &= [\ P_1^m \quad P_1^r \quad P_2^m \quad P_2^r \quad \nu_1 \quad \nu_2\ ]^T \\ u &= [\ u_1 \quad u_2]^T\ . \end{aligned}$$

In this setup, we have

$$P_{22} \in \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^3}\mathcal{R}_p \\ \frac{1}{z^3}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix},$$

which corresponds to a 15ms propagation delay of dynamics between nodes. We assume that the communication delay between controllers is 40ms, and thus we require

$$K \in \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^9}\mathcal{R}_p \\ \frac{1}{z^9}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix} =: \mathcal{S}.$$

$$P = \left[\begin{array}{cccc|cccccc:cc} 1-\frac{\Delta t}{M_1}D_1 & 0 & -\frac{\Delta t}{M_1} & 0 & \frac{\Delta t}{M_1}\gamma_1^m & \frac{\Delta t}{M_1}\gamma_1^r & 0 & 0 & 0 & 0 & -\frac{\Delta t}{M_1} & 0 \\ 0 & 1-\frac{\Delta t}{M_2}D_2 & 0 & -\frac{\Delta t}{M_2} & 0 & 0 & \frac{\Delta t}{M_2}\gamma_2^m & \frac{\Delta t}{M_2}\gamma_2^m & 0 & 0 & 0 & -\frac{\Delta t}{M_2} \\ \Delta t B_{12}\omega^0 & -\Delta t B_{12}\omega^0 & 1-\epsilon\Delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\Delta t B_{21}\omega^0 & \Delta t B_{21}\omega^0 & 0 & 1-\epsilon\Delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \multicolumn{4}{c|}{1000I_{2\times 2} \;\; 0_{2\times 2}} & \multicolumn{6}{c:}{\multirow{2}{*}{$0_{4\times 6}$}} & \multicolumn{2}{c}{0_{2\times 2}} \\ \multicolumn{4}{c|}{0_{2\times 4}} & \multicolumn{6}{c:}{} & \multicolumn{2}{c}{500I_{2\times 2}} \\ \hdashline \multicolumn{4}{c|}{I_{2\times 2} \;\; 0_{2\times 2}} & \multicolumn{6}{c:}{0_{2\times 4} \;\; I_{2\times 2}} & \multicolumn{2}{c}{0_{2\times 2}} \end{array}\right] \tag{15}$$

It is trivially verified that $\mathcal{S}$ is *not* QI under P_{22}. However,

$$\bar{\mathcal{S}}_{QI} = \left\{ \mathcal{S}_1 := \begin{bmatrix} \frac{1}{z^5}\mathcal{R}_p & \frac{1}{z^9}\mathcal{R}_p \\ \frac{1}{z^9}\mathcal{R}_p & \frac{1}{z}\mathcal{R}_p \end{bmatrix},\ \mathcal{S}_2 := \begin{bmatrix} \frac{1}{z}\mathcal{R}_p & \frac{1}{z^9}\mathcal{R}_p \\ \frac{1}{z^9}\mathcal{R}_p & \frac{1}{z^5}\mathcal{R}_p \end{bmatrix} \right\}$$

forms a QI cover of $\mathcal{S}$. The cost obtained when computing the controller with respect to either $\bar{\mathcal{S}}_{QI}$, or $\mathcal{S}$, was 3733. Thus, our heuristic matches the optimal cost obtained when optimizing with respect to the non QI constraint set $\mathcal{S}$. Unfortunately, this comes at a (significant) cost in terms of the controller order: 551 for $\bar{\mathcal{S}}_{QI}$ versus 99 for the non-implementable controller computed with respect to $\mathcal{S}$. We do however believe that it will be possible to achieve similar results without such an explosion in controller order, and this will also be the subject of future work.

V. Conclusion and Future Work

Inspired by this smart-grid applications in which delay patterns may not be QI, we developed an iterative procedure that designs stabilizing sub-optimal controllers for decentralized systems subject to *non-QI* delay patterns. We then applied this procedure to a smart-grid frequency regulation problem, and in this case, observed that our heuristic matched the performance achieved when optimizing with respect to the non-QI constraint set imposed by the communication patterns of the problem.

There are two main avenues for future work. The first is to formalize what a "good" QI cover is for our heuristic, and then to develop an algorithmic procedure for generating one, perhaps exploiting results in [22]. The second is to address scalability issues so as to extend the practicality of this heuristic to large interconnected systems, such as the smart-grid.

References

[1] J. Sandell, N. and M. Athans, "Solution of some nonclassical lqg stochastic decision problems," *Automatic Control, IEEE Transactions on*, vol. 19, no. 2, pp. 108 – 116, apr 1974.

[2] B.-Z. Kurtaran and R. Sivan, "Linear-quadratic-gaussian control with one-step-delay sharing pattern," *Automatic Control, IEEE Transactions on*, vol. 19, no. 5, pp. 571 – 574, oct 1974.

[3] T. Yoshikawa, "Dynamic programming approach to decentralized stochastic control problems," *Automatic Control, IEEE Transactions on*, vol. 20, no. 6, pp. 796 – 797, dec 1975.

[4] P. Varaiya and J. Walrand, "On delayed sharing patterns," *Automatic Control, IEEE Transactions on*, vol. 23, no. 3, pp. 443 – 445, jun 1978.

[5] T. Yoshikawa and H. Kobayashi, "Separation of estimation and control for decentralized stochastic control systems," *Automatica*, vol. 14, no. 6, pp. 623 – 628, 1978. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0005109878900523

[6] B. Kurtaran, "Corrections and extensions to "decentralized stochastic control with delayed sharing information pattern"," *Automatic Control, IEEE Transactions on*, vol. 24, no. 4, pp. 656 – 657, aug 1979.

[7] A. Lamperski and J. C. Doyle, "On the structure of state-feedback lqg controllers for distributed systems with communication delays," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, dec. 2011, pp. 6901 –6906.

[8] ——, "Dynamic programming solutions for decentralized state-feedback lqg problems with comminication delays," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, jun. 2012.

[9] A. Nayyar, A. Mahajan, and D. Teneketzis, "Optimal control strategies in delayed sharing information structures," *Automatic Control, IEEE Transactions on*, vol. 56, no. 7, pp. 1606 –1620, july 2011.

[10] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *Automatic Control, IEEE Transactions on*, vol. 50, no. 12, pp. 1984 –1996, dec. 2005.

[11] M. Rotkowitz, R. Cogill, and S. Lall, "Convexity of optimal control over networks with delays and arbitrary topology," *Int. J. Syst., Control Commun.*, vol. 2, no. 1/2/3, pp. 30–54, Jan. 2010. [Online]. Available: http://dx.doi.org/10.1504/IJSCC.2010.031157

[12] A. Rantzer, "A separation principle for distributed control," in *Decision and Control, 2006 45th IEEE Conference on*, dec. 2006, pp. 3609 – 3613.

[13] A. Gattami, "Generalized linear quadratic control theory," in *Decision and Control, 2006 45th IEEE Conference on*, dec. 2006, pp. 1510 –1514.

[14] A. Lamperski and J. C. Doyle, "Output feedback $\mathcal{H}_2$ model matching for decentralized systems with delays," in *2013 IEEE American Control Conference, To appear.*, sept. 2012.

[15] J. Swigart and S. Lall, "An explicit state-space solution for a decentralized two-player optimal linear-quadratic regulator," in *American Control Conference (ACC), 2010*. IEEE, 2010, pp. 6385–6390.

[16] L. Lessard and S. Lall, "A state-space solution to the two-player decentralized optimal control problem," in *49th Annual Allerton Conference on Communication, Control, and Computing*. IEEE, 2011, pp. 1559–1564.

[17] B. Bamieh and P. G. Voulgaris, "A convex characterization of distributed control problems in spatially invariant systems with communication constraints," *Systems & Control Letters*, vol. 54, no. 6, pp. 575–583, 2005.

[18] P. G. Voulgaris, G. Bianchini, and B. Bamieh, "Optimal h2 controllers for spatially invariant systems with delayed communication requirements," *Systems & control letters*, vol. 50, no. 5, pp. 347–361, 2003.

[19] M. Fardad and M. R. Jovanović, "Design of optimal controllers for spatially invariant systems with finite communication speed," *Automatica*, vol. 47, no. 5, pp. 880–889, 2011.

[20] G. Simard, "Interview with georges simard," in *IEEE Smart Grid – Questions and Answers*, 2012. [Online]. Available: http://smartgrid.ieee.org/questions-and-answers/658-interview-with-georges-simard

[21] S. Sabau and N. C. Martins, "Stabilizability and norm-optimal control design subject to sparsity constraints," *arXiv preprint arXiv:1209.1123*, 2012.

[22] M. Rotkowitz and N. Martins, "On the nearest quadratically invariant information constraint," *Automatic Control, IEEE Transactions on*, vol. 57, no. 5, pp. 1314 –1319, may 2012.

[23] C. Zhao, U. Topcu, and S. H. Low, "Swing dynamics as primal-dual algorithm for optimal load control," in *CaltechAUTHORS*, 2012. [Online]. Available: http://resolver.caltech.edu/CaltechCDSTR:2012.001