ESE 680-004: Learning and Control **Fall 2019**

Lecture 8: A Whirlwind Tour of Robust Control 2

Lecturer: Nikolai Matni *Scribes: Kendall J. Queen*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

This script is the second iteration of instruction on Robust Control with an introduction into structured uncertainty. This document will focus exclusively on the small gain theorem, structured singular value, KYP Lemma, and integral quadratic constraints (IQCs).

1 Introduction

This is a continuation of the discussion on robust stability. We start with a generalized plant seen in Figure 1. This plant is built to have two outputs, controlled and measured, denoted properly below in equations 2 and 3 respectively.

$$x_{t+1} = Ax_t + B_1 w_t + B_2 u_t \quad \textit{State} \tag{1}$$

$$z_t = C_1 x_t + D_{12} u_t \quad \textit{Controlled Output} \tag{2}$$

$$y_t = C_2 x_t + D_{21} u_t \quad \textit{Measured Output} \tag{3}$$

$$P_{ij} = C_i (zI - A)^{-1} B_j + D_{ij} \tag{4}$$

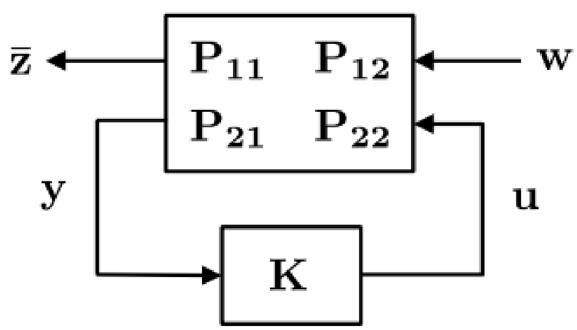

Figure 1: System P with a feedback controller K.

If we take our generalized plant and refer to it as system M and place it in a feedback loop with some uncertainty Δ, which is bounded but stable, we have a new system visualized in Figure 2. This is the Linear Fractional Transform (LTF) representation of uncertainty. This connection is robustly well-connected if $(I - M_{11}\Delta)^{-1}$ exists for all $\Delta \in D$. As defined in class, robust well-connectedness is synonymous with robust stability. Furthermore, we proved that robust stability exists if and only if $I - M\Delta$ is non-singular for all $\Delta \in \Delta_a$.

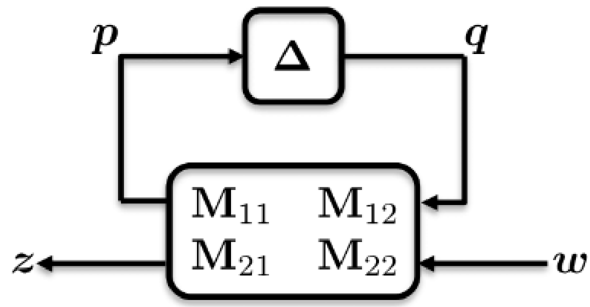

Figure 2: Linear Fractional Transform representation of uncertainty where M is the generalized plant discussed earlier in a feedback loop with uncertainty set Δ.

2 Small Gain Theorem

Theorem 1. *Let $\boldsymbol{Q}$ be a bounded linear operator, $\|\boldsymbol{Q}\|_{H_\infty} < \infty$, and let $\boldsymbol{D} = \{\Delta := \Delta : ||\Delta||_{H_\infty} \leq 1\}$. Then $I - Q\Delta$ is non-singular for all $\Delta \in \boldsymbol{D}$ if and only if $\|\boldsymbol{Q}\|_{H_\infty} < 1$.*

In order for the Small Gain Theorem to apply there are 2 key properties that must hold.

1. Assume M is stable $||M||_{H_\infty} < \infty$
2. $|| \cdot ||_{H_\infty}$ is a sub-multiplicative norm needed to establish that the vector space of bounded linear operators equipped with $|| \cdot ||_{H_\infty}$ norm forms a Banach Algebra.

Key property 2 allows us to take limits and guarantee existence. In the traditional sense, we would assume two stable systems M and Δ are connected via feedback loop as seen in Figure 2. We also assume that we know these systems in regards to gain.

Proof. If: To prove the if direction, we first look at the small gain. At first glance we can apply the multiplicative property and bound $||\boldsymbol{Q}\Delta||$ as seen in equation 5. We know that $||\boldsymbol{Q}||_{H_\infty} < 1$ and $||\Delta|| \leq 1$ by assumption. Therefore, we can simply deduct equation 6.

If $||\boldsymbol{Q}||_{H_\infty} < 1$,

$$||\boldsymbol{Q}\Delta|| \leq ||\boldsymbol{Q}|| \cdot ||\Delta|| \tag{5}$$

$$||\boldsymbol{Q}\Delta|| \leq ||\boldsymbol{Q}|| \cdot ||\Delta|| < 1 \tag{6}$$

With this deduction, we can explicitly construct the $(I - Q\Delta)^{-1}$. Express $(I - Q\Delta)^{-1}$ as a geometric series, seen in equation 7 then prove that this series converges and exists through the triangle inequality and our previous deduction.

$$(I - Q\Delta)^{-1} = \sum_{t=0}^{\infty} (\boldsymbol{Q}\Delta)^t \tag{7}$$

$$|| \sum_{t=0}^{T} (\boldsymbol{Q}\Delta)^t ||_{H_\infty} \leq \sum_{t=0}^{T} ||(\boldsymbol{Q}\Delta)^t||_{H_\infty} \leq \sum_{t=0}^{T} ||\boldsymbol{Q}\Delta||_{H_\infty}^t \tag{8}$$

Equation 8 shows the use of the triangle inequality. We then can simply apply the solution we know for the geometric series as the limit approaches ∞, seen in equation 9 .

$$\lim_{x \to \infty} \sum_{t=0}^{T} ||\boldsymbol{Q}\Delta||_{H_\infty}^t = \frac{1}{1 - ||\boldsymbol{Q}\Delta||} \tag{9}$$

Only if: If $||\boldsymbol{Q}|| \geq 1$, construct a Δ, where $||\Delta|| \leq 1$, such that $I - \boldsymbol{Q}\Delta$ is singular. We start with the observation in equation 10. Normally, the spectral norm of a matrix would equal that of its transpose but in this case it is conjugate tranpose due to $\boldsymbol{Q}$'s complex values. Now forward, we will refer to $\lambda_{max}(\boldsymbol{QQ}^*)$ as λ.

$$||\boldsymbol{Q}||^2_{H_\infty} = ||\boldsymbol{Q}^*||^2_{H_\infty} = \lambda_{max}(\boldsymbol{QQ}^*) \geq 1 \tag{10}$$

Through the definition of an eigenvalue, we know that $\lambda I - \boldsymbol{QQ}^*$ is singular. $\lambda \neq 0$ which allows the scaling operation in equation 11 and does not change its singular status.

$$\frac{1}{\lambda} \cdot (\lambda I - \boldsymbol{QQ}^*) = I - \frac{\boldsymbol{QQ}^*}{\lambda} \tag{11}$$

By writing equation 11 using our definition of λ, we produce equation 12 and claim that $\Delta := \frac{\boldsymbol{Q}^*}{||\boldsymbol{Q}^*||^2_{H_\infty}}$. Then we apply the H_∞ norm to Δ in equation 13 which completes the construction of our counter example.

$$I - \frac{\boldsymbol{QQ}^*}{||\boldsymbol{Q}^*||^2_{H_\infty}} \tag{12}$$

$$||\Delta||_{H_\infty} = \frac{||\boldsymbol{Q}^*||_{H_\infty}}{||\boldsymbol{Q}^*||^2_{H_\infty}} = \frac{1}{||\boldsymbol{Q}^*||_{H_\infty}} \leq 1 \tag{13}$$

Therefore, through this proof we have shown that the condition is both necessary and sufficient. □

3 Scaled Small Gain Test

Small gain is a sufficient but a conservative bound. When working with a block diagonal, defined in equation 14, $||\boldsymbol{M}|| \geq 1$ holds no weight and does not really imply anything. More specifically it does not imply that $I - \boldsymbol{M}\Delta$ is singular for some uncertainty in $\boldsymbol{\Delta}_a$. In order to scale things we use a trick that involves the use of a commutant set, introduced in equation 15.

$$\boldsymbol{\Delta}_a = \boldsymbol{\Delta} : ||\boldsymbol{\Delta}|| \leq 1, \boldsymbol{\Delta} = blkdiag(\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, \ldots, \boldsymbol{\Delta}_d \tag{14}$$

$$\mathcal{G} = \boldsymbol{\Gamma} : \Gamma\boldsymbol{\Delta} = \boldsymbol{\Delta}\Gamma,\ \Gamma^{-1}\ exists \tag{15}$$

The commutant set is the set of operators that commutes with any Δ within the block diagonal, $\boldsymbol{\Delta}_a$, of uncertainty. With the commutant set we plan to prove that $I - \boldsymbol{M}\Delta$ is singular.

3.1 Proof of Small Gain Test with Block

Due to the fact that inverse, Γ^{-1}, shares the same communtant properties we can rewrite $I - \boldsymbol{M}\Delta$, seen in equation 16. This provides us now with a scaled small gain, now slightly altering our proof. Now $I - \boldsymbol{M}\Delta$ is non-singular if and only if $I - \Gamma\boldsymbol{M}\Gamma^{-1}\Delta$ is non-singular. If there exists a Γ such that $||\Gamma\boldsymbol{M}\Gamma^{-1}|| < 1$ then the system is robustly well-connected.

$$\Gamma(I - \boldsymbol{M}\Delta)\Gamma^{-1} = I - \Gamma\boldsymbol{M}\Delta\Gamma^{-1} = I - \Gamma\boldsymbol{M}\Gamma^{-1}\Delta \tag{16}$$

The proof is the same operational steps as the small gain proof seen above in Section 2.

4 Kalman-Yakubovich-Popov (KYP) Lemma

One of the most fundamental tools in systems theory is the Kalman-Yakubovich-Popov (KYP) Lemma. In short, Popov introduced a criterion that gave a frequency condition for stability of a feedback system with a memoryless nonlinearity. In respect to the previous examples, KYP is applied to our feedback uncertainty system and provided a computational test.

Theorem 2. [1] *Give A, B, M, with $det(e^{j\omega}I - A) \neq 0$) for $\omega \in \mathbb{R}$ and (A,B) controllable, the following two statements are equivalent:*

1. $\begin{bmatrix} (e^{j\omega}I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (e^{j\omega}I - A)^{-1}B \\ I \end{bmatrix} \leq 0 \, \forall \omega \in \mathbb{R}$.
2. *There exists a matrix $P \in \mathbb{R}^{nxn}$ such that $P = P^T$ and $M + \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} \leq 0$. The corresponding equivalence for strict inequalities holds even if (A,B) is not controllable.*

At this point, we will prove the sufficiency direction for KYP, where the linear matrix inequality (LMI) implies the norm condition.

Proof. Suppose there exists a P that satisfies the LMI given above. We want to show that the LMI implies the norm condition on the system. The full LMI, equation 17, as it pertains to $\boldsymbol{M}(\mathrm{z}) = C(zI - A)^{-1}B + D$, which is a stable, bounded linear time invariant (LTI) operator. $\boldsymbol{M} = \begin{bmatrix} C \\ D \end{bmatrix} \cdot \begin{bmatrix} C & D \end{bmatrix}$. We then pre and post multiply the LMI.

$$\begin{bmatrix} x_t^T & w_t^T \end{bmatrix} \left(\begin{bmatrix} C \\ D \end{bmatrix} \cdot \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} \right) \begin{bmatrix} x_t \\ w_t \end{bmatrix} \leq 0 \tag{17}$$

After some strategic multiplication and defining our measured output below, equations 18 and 19, we can rewrite our LMI as in equations 20, 21, and 22.

$$x_{t+1} = Ax_t + Bw_t, x_0 = 0 \tag{18}$$

$$z_t = Cx_t + Dw_t \tag{19}$$

$$(Ax_t + Bw_t)^T P(Ax_t + Bw_t) - x_t^T Px_t + (Cx_t + Dw_t)^T (Cx_t + Dw_t) - w_t^T w_t \leq 0 \tag{20}$$

$$x_{t+1}^T Px_{t+1} - x_t^T Px_t + z_t^T z_t < w_t^T w_t \tag{21}$$

$$x_{t+1}^T Px_{t+1} - x_t^T Px_t + ||z_t||_2^2 < ||w_t||_2^2 \tag{22}$$

This becomes a telescoping sum seen in equation 23 and $x_0^T Px_0$ goes to 0 per the definition in equation 18. Then taking the limit as $t \to \infty$ of equation 23 we get equation 24, which leads us into the norm condition as previously described.

$$\sum_{t=0}^{T} ||z_t||_2^2 + x_{t+1}^T Px_{t+1} - x_0^T Px_0 < \sum_{t=0}^{T} ||w_t||_2^2 \tag{23}$$

$$\lim_{t \to \infty} \sum_{t=0}^{T} ||z_t||_2^2 + x_{t+1}^T Px_{t+1} - x_0^T Px_0 < \sum_{t=0}^{T} ||w_t||_2^2 = ||z||_{l_2}^2 < ||w||_{l_2}^2 = \frac{||\boldsymbol{M}w||_{l_2}^2}{||w||_{l_2}^2} < 1, \;\; \forall w, t, l_2 \tag{24}$$

□

[1] [1]

I will point you to [1] for the proof of the sufficient direction, as it is difficult and complicated.

4.1 Structured Uncertainty Proposition, Proposition 8.26

Suppose $\boldsymbol{M}(z) = C(zI - A)^{-1}B + D$ is a stable, bounded LTI operator. Then the following are equivalent:

1. There exists $\Gamma \in \mathcal{G}$ such that $||\Gamma \boldsymbol{M} \Gamma^{-1}|| < 1$.
2. There exists $\Gamma \in \mathcal{G}$ such that Γ is real and positive such that $||\Gamma^{\frac{1}{2}} \boldsymbol{M} \Gamma^{\frac{-1}{2}}|| < 1$
3. There exists $P > 0$ and $\Gamma \in \mathcal{G}$, Γ is real and positive, such that $\begin{bmatrix} A^T PA - P & A^T PB \\ B^T PA & B^T PB - \Gamma \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \Gamma \begin{bmatrix} C & D \end{bmatrix} < 0$

Due to their equivalence, proving one of the pieces of the proposition will hold true for the rest of them. Thereby, I will select the second condition to manipulate. Equation 25 below is an expansion of M and shows that B, C, and D will be scaled by the $\Gamma^{\frac{1}{2}}$ and $\Gamma^{\frac{-1}{2}}$ terms. Then we take the those scaled terms and plug them into a structure similar to equation 17 with pre multiplication of $\begin{bmatrix} I & 0 \\ 0 & \Gamma^{\frac{1}{2}} \end{bmatrix}$ and post multiplication of $\begin{bmatrix} I & 0 \\ 0 & \Gamma^{\frac{1}{2}} \end{bmatrix}$. This is expounded in equation 26.

$$\Gamma^{\frac{1}{2}} M \Gamma^{\frac{-1}{2}} = \Gamma^{\frac{1}{2}} C(zI - A)^{-1} B \Gamma^{\frac{-1}{2}} + \Gamma^{\frac{1}{2}} D \Gamma^{\frac{-1}{2}} \tag{25}$$

$$\begin{bmatrix} I & 0 \\ 0 & \Gamma^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x_t^T & w_t^T \end{bmatrix} \left(\begin{bmatrix} C \\ D \end{bmatrix} \cdot \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^T PA - P & A^T PB \\ B^T PA & B^T PB \end{bmatrix} \right) \begin{bmatrix} x_t \\ w_t \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \Gamma^{\frac{1}{2}} \end{bmatrix} \leq 0 \tag{26}$$

Since we know that the setup from equation 17 is a positive definite matrix which is being sandwiched by symmetric positive matrices, we can assume that the conditions do not change. This then refers back to the idea of a scaled small gain test previously discussed in Section 3.

5 Structured Singular Value

The Structured singular value addresses the a problem where there is more structure in the uncertainty of the system. In the slides, we spoke about scalar and matrix uncertainties with restrictions which would apply the idea of a structure singular value. This is necessary because even with a small gain or scaled small gain test is still very conservative in bounding.

Definition 1. *Given an uncertainty set $\boldsymbol{\Delta}$, the structured singular value of an operator $\boldsymbol{M}$ is*

$$\mu(\boldsymbol{M}, \boldsymbol{\Delta}) := \frac{1}{inf\{||\Delta|| : \Delta \in \boldsymbol{\Delta} \; and \; (I - \Delta \boldsymbol{M}) is \; singular\}}$$

,

if the infinmum is finite or defined. Otherwise $\mu(\boldsymbol{M}, \boldsymbol{\Delta}) = 0$.

The structured singular value can be applied to structured uncertainty examples such as the following.

1. Single Uncertainty: $\boldsymbol{\Delta}_s = \left\{ \delta I : \delta \in \mathcal{C}, \; 0 < |\delta| \leq 1 \right\}$
2. Block Diagonal Uncertainty: $\boldsymbol{\Delta} \in \mathcal{C}^{nxn} : \; \boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Delta}_1 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{\Delta}_r \end{bmatrix}$

I use $\boldsymbol{\Delta}$ to define a certainty set in the above equations. In item 1, we see an uncertainty set where the diagonal element, δI, is a one singular value between 0 and 1. This makes the computation of the structured singular value pretty straightforward.

$$||\boldsymbol{\Delta}|| = \sigma_{max}(\boldsymbol{\Delta}) = |\delta| \leq 1 \, for \; all \; \boldsymbol{\Delta} \in \boldsymbol{\Delta}_s$$

In words, we are saying that any uncertainty constrained to the set of $\boldsymbol{\Delta}_s$ then we can find the eigenvalue, λ of operator $\boldsymbol{M}$.

$$|I - A\boldsymbol{\Delta}| = |I - \delta \boldsymbol{M}| = 0 = \delta|\delta^{-1}I - \boldsymbol{M}|$$

We then set the eigenvalue $\lambda = \delta^{-1}$ and find the new solution below. This proves that λ is an eigenvalue of $\boldsymbol{M}$ by the eigenvalue equation.

$$|\lambda I - \boldsymbol{M}| = 0$$

This also raised the question of bounds for δ and why it cannot be 0. If $\delta = 0$ then it would invalidate the first step, making the determinant, $|I - \delta \boldsymbol{M}|, = 1$.
The second example is dealing with a an uncertainty set that is a matrix of blocks, $\boldsymbol{\Delta}_x$, that each of a scalars along the diagonals as $\boldsymbol{\Delta}_i = \delta_i I$. If you could imagine, this process has the same cruxes as the first example you just have to proceed for every block in the uncertainty set.
In general it is about NP-hard to compute the structured singular value, but it is very well-studied. In addition, good computationally tractable upper and lower bounds exist to better understand the

5.1 S Procedure

The S-procedure is used when your uncertainty is not linear, exemplified in equation 27 where we would want to prove stability. In this case, we would use the Lyapunov function, $V(x) = x^T Px$, $P > 0$, to prove stability. Note the lipschitz bound placed on $||g(x_t)||$.

$$x_{t+1} = Ax_t + g(x_t), \; ||g(x_t)||_2 \leq \gamma ||x_t||_2 \tag{27}$$

More specifically we would need $V(x_{t+1}) - V(x_t) \leq -\epsilon V(x_t)$. Equation 27 shows the difference between our next state and current state cannot be any larger than some scaled version of the current state, where $\epsilon > 0$. This is a condition necessary for exponential stability for some $\epsilon > 0$. This can be equivalently written in the form in equations 28 and 29 by plugging in the dynamics and manipulating the inequality. Note the substitution of $g(x_t) = z_t$.

$$(Ax_t + z_t)^T P(Ax_t + z_t) - x_t^T Px_t < -\epsilon(x_t^T Px_t), \;\; \forall t \, where \; z_t = g(x_t) \tag{28}$$

$$(Ax_t + z_t)^T P(Ax_t + z_t) - x_t^T Px_t + \epsilon(x_t^T Px_t) < 0 \Rightarrow (Ax_t + z_t)^T P(Ax_t + z_t)(1 - \epsilon)x_t^T Px_t < 0 \tag{29}$$

$$\begin{bmatrix} x_t \\ z_t \end{bmatrix}^T \begin{bmatrix} A^T PA - (1-\epsilon)P & A^T P \\ PA & P \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} \leq 0 \tag{30}$$

Equation 30 shows a matrix representation of equation 29. Notice the pre and post multiply that is similar to what we see in the KYP Lemma in Section 4. At this point, we would only need to find a P that satisfies the inequality $P > 0$. By renaming the inequality in equation 30 as $*$, we need to find $P > 0$ such that $(*)$ $\forall z_t = g(x_t)$. In order to encode this information we can rewrite as seen in equation 31.

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0 \Rightarrow \gamma^2 ||x||^2 \geq ||z||^2 \tag{31}$$

Equation 31 enforces that if z is produced by g(x), which satisfies the lipschitz bound, then the inequality $*$ is also satisfied. The use of the S-procedure allows us to prove that this holds true.
In an abstract sense, $v^T F_1 v \geq 0 \Rightarrow v^T F_0 v \geq 0$, where $F_i = F_i^T$. When referring to the problem stated above, $v^T F_1 v \geq 0$ is a reference to the constraint, lipschitz bound, from above and $v^T F_0 v \geq 0$ is the stability constraint, which is implied from the constraint.
To prove the sufficient condition:

Proof. $\exists \tau \geq 0$ such that $F_0 \geq \tau F_1$, in the positive semi-definite sense.

$$z^T F_1 z \geq 0 \Rightarrow z^T F_0 z \geq \tau z^T F_1 z \geq 0 \tag{32}$$

□

This is a necessary and an exact condition. The necessary direction is very hard to prove, but can be raised through proving if $\exists u$ such that $u^T F_1 u > 0$.

5.2 Lossless S-Procedure

In reference to the example presented in Section 5.1, specifically equation 30, if we set $\epsilon = 0$, we would have $v^T F_1 v \geq 0$, where $v \neq 0$, which implies $v^T F_0 v > 0$. This would bring us to the same result where the implication is true if and only if $F_0 > \tau F_1$ as long as the constraint qualification, $\exists u$ such that $u^T F_1 u > 0$, holds. We refer to this as a lossless S-procedure.
Therefore to complete our example and link our bounds and stability constraint we proceed as below.
True if and only if $\exists \tau \geq 0$ and $P > 0$ such that

$$\tau \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} A^T P A - (1-\epsilon)P & A^T P \\ PA & P \end{bmatrix} < 0 \tag{33}$$

5.3 Lossy S-Procedure

The Lossy S-procedure is best exemplified when dealing with multiple quadratic forms. For example, we let $F_0, ..., F_k$ be symmetric matrices and we want a sufficient condition such that $v^T F_1 v > 0$, ... , $v^T F_k v > 0$ $\Rightarrow v^T F_0 v \geq 0$. This condition brings up the question of when does nonnegativity of a set of quadratic forms imply nonnegativity of another set. A simple sufficient condition for this is to suppose there are $\tau_1, ..., \tau_k \geq 0$, with $F_0 \geq \tau_1 F_1 + ... + \tau_k F_k$. This solidifies our original sufficient condition, with no necessary conditions known, meaning that more assumption are needed but lossless versions do exist. This is a Lossy S-Procedure. In fact what we have just designed is a special case of an Integral Quadratic Constant (IQC), which extends t the frequency domain.

6 IQC

This a relatively brief introduction to IQCs. To gather a more detailed view in IQCs and how they could benefit your work I would recommend [2] and [3].

The term integral quadratic constraint (IQC) was introduced in [3], where they explored continuous-time dynamical systems which included the constraints present in integrating quadratic functions. This is now known as a popular technique in control theory to understand the behavior of partially known components, similar to the idea of uncertainty that we have been discussing. We will be adapting the classical IQC theory for algorithm analysis using discrete-time dynamical systems.

The general idea of using IQC's is to replace the troublesome or uncertain components of an interconnected dynamical system. By placing a quadratic constraint on its inputs and outputs, we can understand all possible instances of the component and certify that the system performs as desired.

To relate back to class, we can use the Lipschitz bound to characterize constraints on the input and output pairs of $(\boldsymbol{y},\ \boldsymbol{u})$: $\boldsymbol{u} = \Delta(\boldsymbol{y})$. In this case, we do not exactly know Δ, but we do assume that we know something of the constraints it imposes on the pair $(\boldsymbol{y},\ \boldsymbol{u})$. Explicitly, let's assume Δ is

1. Static and memoryless: $\Delta(y_0, y_1, ...) = (g(y_0), g(y_1), ...)$ for some g: $\mathcal{R}^d \to \mathcal{R}^d$.
2. g is Lipschitz bounded: $||g(y_1) - g(y_2)|| \leq \mathcal{L}||y_1 - y_2||$ for all $y_1,\ y_2 \in \mathcal{R}^d$.

if $\boldsymbol{y} = (y_0, y_1, ...)$ which is a sequence of vectors in $\mathcal{R}^d$ and $\boldsymbol{u} = \Delta(y)$, the output of the unknown function, then Property 2 implies that $||u_k - u_*|| \leq ||y_k - y_*||$ for all k, where (y_*, u_*) is any pair of vectors satisfying $u_* = g(y_*)$. The matrix form can be seen in 34.

$$\begin{bmatrix} y_k - y_* \\ u_k - u_* \end{bmatrix}^T \begin{bmatrix} L^2 I_d & 0_d \\ 0_d & -I_d \end{bmatrix} \begin{bmatrix} y_k - y_* \\ u_k - u_* \end{bmatrix} \geq 0 \ \ for\ k = 0, 1, ... \tag{34}$$

The quadratic coupling of (y, u) is pointwise, meaning that it holds as sparate quadratic constraints on each (y_k, u_k). In order to generalize this idea, and couple different k values you must introduce auxiliarty sequences $\zeta, z \in l_{2e}$, with a map Ψ characterized by matrices $(A_\Psi, B_\Psi^y, B_\Psi^u, C_\Psi, D_\Psi^y, D_\Psi^u)$ and the recursion below.

$$\zeta_0 = \zeta_* \tag{35}$$

$$\zeta_{k+1} = A_\Psi \zeta_k + B_\Psi^y y_k + B_\Psi^u u_k \tag{36}$$

$$z_k = C_\Psi \zeta_k + D_\Psi^y y_k + D_\Psi^u u_k \tag{37}$$

This creates an affine map $\boldsymbol{z} = \Psi(\boldsymbol{y}, \boldsymbol{u})$ and with an assumed reference point (y_*, u_*) we can continue below.

$$\zeta_* = A_\Psi \zeta_* + B_\Psi^y y_* + B_\Psi^u u_* \tag{38}$$

$$z_* = C_\Psi \zeta_* + D_\Psi^y y_* + D_\Psi^u u_* \tag{39}$$

We then ensure that equations 38 and 39 have unique solutions (ζ_*, z_*) for any choice (y_*, u_*) through requiring $\rho(A_\Psi) < 1$. Equation 40 characterizes our $(\boldsymbol{y},\boldsymbol{u})$ pair in terms of the quadratic constraints, this in effect will allow us to analyze a system where our uncertainty is abstracted away. We can usually assume that Ψ is LTI and use equation 40 and Parseval find the equivalent representation found in 41.

$$z^* F z = \begin{bmatrix} y \\ u \end{bmatrix}^* \Psi^* F \Psi \begin{bmatrix} y \\ u \end{bmatrix} \geq 0 \tag{40}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \begin{bmatrix} y(e^{j\omega}) \\ u(e^{j\omega}) \end{bmatrix}^* \Psi(e^{j\omega})^* F \Psi(e^{j\omega}) \begin{bmatrix} y(e^{j\omega}) \\ u(e^{j\omega}) \end{bmatrix} \geq 0 \tag{41}$$

[2] gives a detailed description of the different types of IQCs that can be utilized.

7 Computational Example

In this section, I will present an adapted example from MATLAB LMI Control Toolbox where we will verify the stability of an interconnected control system by using the IQCβ tool box, adapted from the ideas and topics covered with robust stability and IQCs seen in the lecture notes of Ulf Jonsson [4] and extensions of Alexandre Megretski. You can find the tutorial and toolbox at [5].

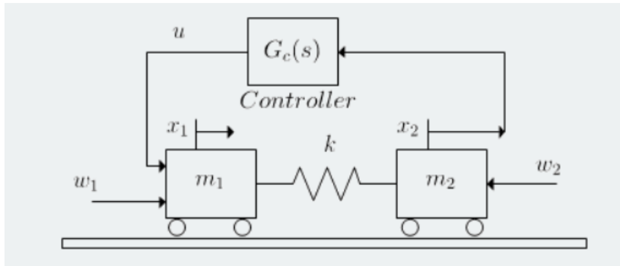

Figure 3: Two-mass-one-spring system

The problem we will explore is the two-mass-one-spring system as seen in Figure 3. The dynamics of the system are described in equation 42, for $m_1 = m_2 = 1$. In this equation, the state vector x contains the position and velocity of the masses m_1 and m_2.

$$\dot{x} = Ax + B_1(u + w_1) + B_2 w_2 \tag{42}$$

Below are the given definitions of matrices $A, B_1,\ and\ B_2$:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Matrix A depends affinely on k. So by letting $k = 1.25 + \frac{3}{4}\bar{k}$, A can be further expressed as $A = A_0 + A_1 \bar{k} A_2^T$, where A_0, A_1, and A_2 are constant matrices and is an unknown constant in the range of [-a,a].

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.25 & 1.25 & 0 & 0 \\ 1.25 & -1.25 & 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 \\ 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix},\ A_2 = \begin{bmatrix} \frac{3\sqrt{2}}{4} \\ -\frac{3\sqrt{2}}{4} \\ 0 \\ 0 \end{bmatrix}$$

Thereby when $a = 1$, the spring coefficient k will be in the range of [0.5, 2], which comes from the original control design specifications of the original problem. Also proposed from the original problem is a 4th order stabilizing controller $u = C_c(sI - A_c)^{-1} B_c x_2$.

$$A_c = \begin{bmatrix} 0 & -0.7195 & 1 & 0 \\ 0 & -2.9732 & 0 & 1 \\ -2.5133 & 4.8548 & -1.7287 & -0.9616 \\ 1.0063 & -5.4097 & -0.0081 & 0.0304 \end{bmatrix},\ B_c = \begin{bmatrix} 0.720 \\ 2.973 \\ -3.37 \\ 4.419 \end{bmatrix},\ C_c^T = \begin{bmatrix} -1.506 \\ 0.494 \\ -1.738 \\ -0.932 \end{bmatrix}$$

The mass-spring control system is expressed as a block diagram in Figure 4, where B_1, B_2, and C are $[0010]^T$, $[000-1]^T$ and $[0100a]$ respectively.

Next I will compute an estimate of the energy gain from $[w1, w2]$ to x_2. Initially estimating the gain from some external disturbance to any internal signal in the system is the best way to check for stability. A finite gain implies stability. This will all be defined in a MATLAB workspaces and will incorporate IQCβ and the MATLAB Control Systems Toolbox.

Start with initializing the abstract IQC-environment, which is necessary for the use of IQCβ. Then I define the signals and design how they relate to the system. At this stage, the uncertainty is $\bar{k}$, therefore that block will be removed and will later be replaced by set-valued functions defined by IQCs. The basic signals (w_1, w_2, w_3) cannot be derived from any other signals through a constant LTI transformation. They are considered external disturbances. Figure **??** shows the the breaking of loop found in Figure 4 that described every signal as a LTI transformation of the other signals and introduced an extra basic signal u. We then

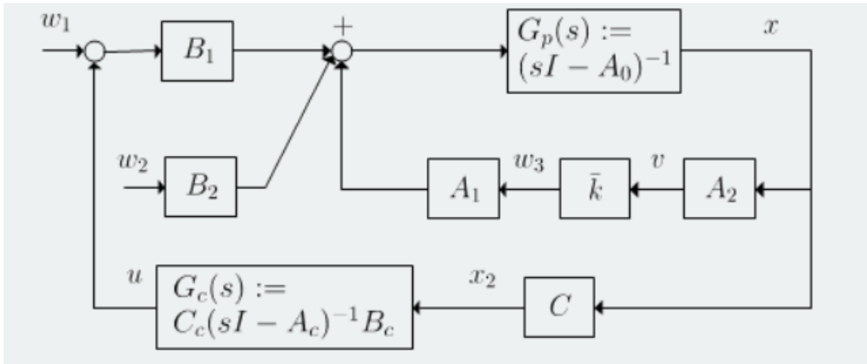

Figure 4: Block diagram of the two-mass-one-spring system

define x, x_2, and u_c as LTI transformations of basic signals including u. I then define the 4 basic signals (w_1, w_2, w_3, and u) as $u_c == u$.

```
abst_init_iqc;  %Initialize IQC-environment

%Define basic signals
w1=signal;
w2=signal;
w3=signal;
u=signal;
```

Now we relate the remaining signals (x, x_2, v, u_c) in the system to the basic signals we just defined.

```
%Relate basic signals to the remaining signals
x=Gp*(A1*w3+B1*(u+w1)+B2*w2);
v=transpose(A2)*x;
x2=x(2); %Also x2 = C*x
uc=Gc*x2;
```

Now we design the IQC. More explicityly we must relate $w3$ and v with a set-valued function. Because $|\bar{k}| \leq a$ we know that

$$\int_0^\infty (a^2|v(t)|^2 - |w3(t)|^2)dt \geq 0$$

The inequality remains valid with the multiplication of any scalar larger than 0, so the following parameterized inequality holds for all pairs of $(w3.v)$ which satisfy $w3 = \bar{k}v$. In this case $M \geq 0$.

$$\int_0^\infty M \cdot (a^2 \cdot v^2 - w_3^2)dt \geq 0$$

```
%Define IQC
M=variable; %creates LMI toolbox variable
M > 0;
v'*(a 2*M)*v-w3'*M*w3>0; %descibes the IQC using toolbox
u == uc;
```

Now we will execute the optimizer to estimate the L2-gain from [w1; w2] to x2. The iqc_gain_tbx.m script will form an optimization problem which corresponds to the worst case L2-gain estimation problem. This formed as a semi-definite program (SDP). Then the LMI Control Toolbox provides the genetic SDP solver to solve the optimization problem. More explicity, the solver will choose the variable parameters M and g so that

$$\int_0^\infty (g(w_1^2 + w_2^2) - x_2^2)dt > \int_0^\infty M(a^2 \cdot v^2 - w_3^2)dt$$

holds for all non-zero L2 signals. The solver will also attempt to minimize g.

```
%run solver to find values for g and M
g = iqc_gain_tbx([w1;w2],x2)
value_iqc(M)
```

Below is output from Matlab Terminal.

```
iqc_extract: processing the abst log information ...
  Processing cst, var, and lin, counting ...
    scalar inputs:  4
    states:         26
    simple q-forms: 2
  Processing signals and quadratic forms ...
    LMI #1    size = 1    states: 0
  iqc_extract done OK

iqc_gain_tbx ...
  defining the original variables ...
  defining the non-KYP LMIs ...
  defining the KYP LMIs ...
  Solving with 353 decision variables ...

 Solver for linear objective minimization under LMI constraints

 Iterations   :    Best objective value so far

     1
     2
     3
     4
     5
     6
     7
     8
     9
    10
    11
    12
    13
    14
    15
    16
    17
    18
    19
    20
    21
    22
    23
    24
* switching to QR
    25
    26
    27                 389.671972
    28                 359.120709
    29                 359.120709
    30                 334.780009
    31                 334.780009
```

```
    32                   334.780009
    33                   308.604259
    34                   308.604259
    35                   308.604259
    36                   308.604259
    37                   308.604259
    38                   308.604259
    39                   308.604259
    40                   308.604259
    41                   308.604259
    42                   308.604259
    43                   308.604259

 Result:  feasible solution
          best objective value:   308.604259
          f-radius saturation:  89.312% of R =  1.00e+09
 Termination due to SLOW PROGRESS:
          the objective was decreased by less than
          1.000% during the last 10 iterations.


g =

   17.5671


ans =

   94.5067
```

The final result shows 94.5067. This shows that the value of M need to achieve L2 gain estimation (g) 17.5671 is 94.5067. According to the tutorial the final result should show 117.9644, which means the value of M to obtain L2-gain estimation (g) 17.4074 is 117.9644.

References

[1] Anders Rantzer. On the kalman—yakubovich—popov lemma. *Systems Control Letters*, 28(1):7 – 10, 1996.

[2] Laurent Lessard, Benjamin Recht, and Andrew Packard. Analysis and design of optimization algorithms via integral quadratic constraints, 2014.

[3] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 42(6):819–830, June 1997.

[4] Ulf Jonsson. Lectures on input-output stability and integral quadratic constraints. University Lecture, 2001.

[5] Iqc toolbox: A matlab toolbox for robust stability and performance analysis. http://140.117.167.218/. Accessed: 2019-10-27.