ESE 680-004: Learning and Control **Fall 2019**

Lecture 4: Finite-data guarantees and data-dependent bounds

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1 Recap

Recall that we were analyzing the scalar dynamical system

$$\begin{aligned} x_{t+1} &= ax_t + u_t + w_t \\ x_0 &= 0 \end{aligned}$$

where $w_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_w^2)$, $u_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_u^2)$, and a, x, u, $w \in \mathbb{R}$.

We ran N experiments and then solved for the estimate $\hat{a}$ via least-squares

$$\begin{aligned} \hat{a} &= \arg\min_a \sum_{i=1}^{N} (x_{T+1}^{(i)} - ax_T^{(i)} - ux_T^{(i)})^2 \\ &= a + \frac{|\sum_{i=1}^{N} x_T^{(i)} w_T^{(i)}|}{\sum_{i=1}^{N} (x_T^{(i)})^2} \\ &\coloneqq a + e_N \end{aligned}$$

To analyze e_N, where,

$$|e_N| = \frac{|\sum_{i=1}^{N} x_T^{(i)} w_T^{(i)}|}{\sum_{i=1}^{N} (x_T^{(i)})^2} \tag{1}$$

We divide the problem into two problems:

1. Find a high probability upper bound on numerator
2. Find a high probability lower bound on denominator

2 Lower bound on denominator, $\sum_{i=1}^{N} (x_T^{(i)})^2$

Deriving the bounds requires us to assume that we utilize only the last two data-points from each trial in order to simplify the analysis.

How to utilize all the data (the single trajectory case) and derive high probability bounds is presented in a subsequent lecture.

Proposition 1. *Fix a failure probability $\delta \in (0,1]$, and let $N \geq 32\log(\frac{1}{\delta})$. Then with probability at least $1-\delta$, we have that*

$$\sum_{i=1}^{N}(x_T^{(i)})^2 \geq \sigma_x^2 \frac{N}{2} \tag{2}$$

where, $\sigma_x^2 := (\sigma_w^2 + \sigma_u^2) + \sum_{t=0}^{T-1} a^{2t}$

Proof.

From previous examples, we know that for $X \sim \mathcal{N}(0,1)$, X^2 is sub-exponential with parameters (4,4). We saw that if X_1 and X_2 are sub-exponential with parameters $(\nu_i^2, \alpha_i)_{i=1,2}$, then $X_1 + X_2$ is sub-exponential with parameters $(\nu_1^2 + \nu_2^2, \max \alpha_1, \alpha 2)$.

We let $\sigma_x^2 = \mathbb{E}x_T^2$ (the variance of our state). Since X_T is Gaussian, dividing by it's standard deviation yields a standard normal distribution, $\frac{x_T}{\sigma_T} \sim \mathcal{N}(O,1)$.

From this, we know that $\frac{x_T^2}{\sigma_T^2}$ is sub-exponential(4,4) with mean 1 and $\sum_{i=1}^{N} \frac{(x_T^2)^(i)}{\sigma_x^2}$ is sub-exponential (4N,4) with mean N.

We apply $\mathbb{P}[X - \mathbb{E}X \geq t] \leq \exp\left(\frac{-t^2}{2\nu^2}\right)$ if $0 \leq t \leq \nu^2/\alpha$ to $\sum_{i=1}^{N} \frac{(x_T^2)^{(i)}}{\sigma_x^2}$ and derive the bounds with respect to N.

$$\begin{aligned}
\mathbb{P}\left[\sum {x_T^{(i)}}^2 - N\sigma_x^2 \leq -t\right] &= \mathbb{P}\left[\sigma_x^2\left[\sum \frac{(x_T^{(i)})^2}{\sigma_x^2} - N\right] \leq t\right] \\
&= \mathbb{P}\left[\sum \frac{(x_T^{(i)})^2}{\sigma_x^2} - N \leq \frac{-t}{\sigma_x^2}\right] \leq \exp\left(\frac{-t}{8N\sigma_x^4}\right) \qquad \forall t \leq N\sigma_x^2
\end{aligned}$$

We invert this expression by setting $\delta = RHS$, and solving for t

$$t = \sigma_x^2\sqrt{8N\log(1/\delta)} \leq N\sigma_x^2 \qquad \text{(assumed } N \geq 32\log(1/\delta))$$

So we've shown that:

$$\sum \left(x_T^{(i)}\right)^2 \geq \underbrace{N\sigma_x^2}_{\mathbb{E}\sum x_T^2} - \underbrace{\sigma_x\sqrt{8N\log(1/\delta)}}_{\text{error term}}$$

Since

$$N \geq 32\log(1/\delta) \geq \sigma_x^2 \frac{N}{2}$$

■

Comments: Why is $\frac{x_T}{\sigma_T} \sim \mathcal{N}(O,1)$? Nice things happen when linear systems are driven by Gaussian noise.

1. If, $x_{t+1} = ax_t + u_t + w_t$, $x_0 = 0$, $w_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,\sigma_w^2)$, $u_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,\sigma_u^2)$,

   we first compute the first moment,

$$\begin{aligned}
X_1 &= AX_0 + Bu_0 + w_0 = Bu_0 + w_0 \\
\mathbb{E}[X_1] &= \mathbb{E}(Bu_0 + w_0) = B\mathbb{E}[u_0] + E[w_0] = 0
\end{aligned}$$

2. We compute the second moment $\sigma_x^2 = \mathbb{E}[X_T^2]$ using induction.

$$\begin{aligned} X_T &= aX_{T-1} + u_{T-1} + w_{T-1} \\ X_{T-1} &= aX_{T-2} + u_{T-2} + w_{T-2} \\ &\vdots \\ X_1 &= aX_0 + u_1 + w_1 \\ &\downarrow \\ X_T &= a^T x_0 + \sum_{t=0}^{T-1} a^{t-1}(u_{T-t} + w_{T-t}) \end{aligned}$$

Expanding terms and using linearity of expectation we obtain,

$$\mathbb{E}[X_T^2] = \mathbb{E}[(\sum_{t=0}^{T-1} a^{t-1}(u_{T-t} + w_{T-t})) \cdot (\sum_{t=0}^{T-1} a^{t-1}(u_{T-t} + w_{T-t}))]$$

. Since $w_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_w^2)$, $u_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_u^2)$,

$$\begin{aligned} \mathbb{E}[w_t w_\tau] &= \begin{cases} 0 & \text{if } t \neq \tau \\ 1 & \text{if } t = \tau \end{cases} \\ \mathbb{E}[u_t w_\tau] &= 0 & \forall t, \tau \end{aligned}$$

Therefore,

$$\mathbb{E}[X_T^2] = \sum_{t=0}^{T-1} a^{t-2}(\underbrace{\mathbb{E}[u_t^2]}_{\sigma_u^2} + \underbrace{\mathbb{E}[w_t^2]}_{\sigma_w^2})$$

3 Upper bound on numerator, $|\sum_{i=1}^N x_T^{(i)} w_T^{(i)}|$

In order to find the upper bound we first state the following proposition;

Proposition 2. *Fix a failure probability $\delta \in (0, 1]$, and let $N \geq 2\log(\frac{1}{\delta})$. Then with probability at least $1-\delta$, we have that*

$$\|\sum_{i=1}^N x_T^{(i)} w_T^{(i)}\| \leq 2\sigma_x \sigma_w \sqrt{N \log(2/\delta)} \tag{3}$$

where, $\sigma_x^2 := (\sigma_w^2 + \sigma_u^2) + \sum_{t=0}^{T-1} a^{2t}$

Before we go into the proof, observe the following:

- In the previous section, we showed that the denominator $\gtrsim N$. From the proposition, the numerator $\lesssim \sqrt{N}$. Therefore, $e_N \lesssim \frac{1}{\sqrt{N}}$.
- This relates to a tradeoff between how many trials vs. how long I can run each trial.

Proof. *(sketch)* From previous examples, we know that for $X, W \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$, XW is sub-exponential with parameters $(2, \sqrt{2})$. We saw that if X_1 and X_2 are sub-exponential with parameters $(\nu_i^2, \alpha_i)_{i=1,2}$, then $X_1 + X_2$ is sub-exponential with parameters $(\nu_1^2 + \nu_2^2, \max \alpha_1, \alpha 2)$.

Letting $\sigma_x^2 = \mathbb{E}x_T^2$, $\frac{x_T}{\sigma_T}$ and $\frac{w_T}{\sigma_T} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(O, 1)$.

From this, we know that $\frac{x_T}{\sigma_T}\frac{w_T}{\sigma_T}$ is sub-exponential$(2, \sqrt{2})$ and $\sum_{i=1}^N \frac{(x_T)^{(i)}(w_T)^{(i)}}{\sigma_x \sigma_w}$ is sub-exponential $(2N, \sqrt{2})$ with mean N.

We apply $\mathbb{P}[X - \mathbb{E}X \geq t] \leq \exp\left(() \frac{-t^2}{2\nu^2}\right)$ if $0 \leq t \leq \nu^2/\alpha$ to $\sum_{i=1}^N \frac{(x_T)^(i)(w_T)^(i)}{\sigma_x \sigma_w}$ similar to before.

■

From Proposition 1 and Proposition 2, we obtain the following theorem:
Theorem 3.1. *Fix a failure probability* $\delta \in (0, 1]$ *and let* $N \geq 32\log(\frac{2}{\delta})$. *Then with probability* $1 - \delta$,

$$|e_N| = \frac{|\sum_{i=1}^N x_t^{(i)} w_t^{(i)}|}{\sum_{i=1}^N (x_T^{(i)})^2} \leq 4\frac{\sigma_w}{\sigma_x}\sqrt{\frac{\log(2/\delta)}{N}}$$

Proof. The proof is based on the choice of N where we can show that the numerator is big with probability $\leq \delta/2$ and that the denominator is small with probability $\leq \delta/2$, then union bound. ■

4 Full state system ID with IID trials

4.1 Overview

Let us now examine the full state linear time invariant system:

$$x_{t+1} = Ax_t + Bu_t + w_t$$

with $w_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_w^2 I_n)$, $x \in \mathbb{R}^n$, control input $u_t \in \mathbb{R}^p$, disturbance $w \in \mathbb{R}^n$

Our goal is to identify the unknown (A, B). To do this, we run N experiments over a horizon of $T + 1$ steps, injecting random inputs $u_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_u^2 I_p)$ to generate the set $\{x^(i)_{0:T+1}, u^(i)_{0:T+1}\}_{i=1}^N$.

We solve the OLS: $(\hat{A}, \hat{B}) = \arg\min_{(A,B)} \sum_{i=1}^N \|x_{T+1}^{(i)} - Ax_T^{(i)} - Bu_T^{(i)}\|$. Notice, we are only summing the last two data points so that the terms in the sum are i.i.d.

We wish to characterize the convergence rate properties of the estimates: $(\hat{A}, \hat{B}) \to (A, B)$.

4.2 Notation

In order to simplify the arguments following, we first define some notation:

Let

$$X_N := \begin{bmatrix} (x_{T+1}^{(1)})^\top \\ \vdots \\ (x_{T+1}^{(N)})^\top \end{bmatrix}, Z_N := \begin{bmatrix} (x_{T+1}^{(1)})^\top, (u_{T+1}^{(1)})^\top \\ \vdots \\ (x_{T+1}^{(N)})^\top, (u_{T+1}^{(N)})^\top \end{bmatrix}, W_N := \begin{bmatrix} (w_{T+1}^{(1)})^\top \\ \vdots \\ (w_{T+1}^{(N)})^\top \end{bmatrix}$$

Then we can rewrite

$$\begin{aligned} \begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix}^\top &= \arg\min_{(A,B)} \sum_{i=1}^N \|x_{T+1}^{(i)} - Ax_T^{(i)} - Bu_T^{(i)}\| \\ &= \arg\min_{(A,B)} \|X_N - Z_N[A \quad B]^\top\|_F^2 \\ &= [A \quad B]^\top + (Z_N^\top Z_N)^{-1} Z_N^\top W_N \end{aligned}$$

where,

$$
\begin{aligned}
Z_N^\top W_N &= \sum_{i=1}^{N} \begin{bmatrix} x_T^{(i)} \\ u_T^{(i)} \end{bmatrix} (w_T^{(i)})^\top \\
Z_N^\top Z_N &= \sum_{i=1}^{N} \begin{bmatrix} x_T^{(i)} \\ u_T^{(i)} \end{bmatrix} \begin{bmatrix} x_T^{(i)} \\ u_T^{(i)} \end{bmatrix}^\top
\end{aligned}
$$

Notice that $Z_N^\top Z_N$ acts as the *denominator* and $Z_N^\top W_N$ acts as the *numerator*.

From this, we can define the error in (A, B), i.e., spectral norm bounds, as

$$
\begin{aligned}
E_A &:= (\hat{A} - A)^\top = \begin{bmatrix} I_{n_x} & 0_{n_x \times n_u} \end{bmatrix} (Z_N^\top Z_N)^{-1} Z_N^\top W_N \\
E_B &:= (\hat{B} - B)^\top = \begin{bmatrix} 0_{n_x \times n_u} & I_{n_u} \end{bmatrix} (Z_N^\top Z_N)^{-1} Z_N^\top W_N
\end{aligned}
$$

.

4.3 Error bounds for E_A and E_B

We will focus on E_A. Similar arguments hold for E_B.

We know that $w_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_w^2 I_n)$.

For, $\begin{bmatrix} x_T^{(i)} & u_T^{(i)} \end{bmatrix}^\top$,

Through an inductive argument, we can derive that

$$
X_T = \sum_{t=1}^{T} A^{t-1}(Bu_{T-t-1} + w_{T-t-1})
$$

By the linearity of expectation

$$
\mathbb{E} X_T = 0
$$

similarly,

$$
\mathbb{E} X_T X_T^\top = \sum_{t=1}^{T} A^{t-1} B B^\top (A^{t-1})^\top \sigma_u^2 + \sum_{t=1}^{T} A^{t-1} (A^{t-1})^\top \sigma_w^2
$$

Therefore,

$$
\begin{bmatrix} x_T^{(i)} \\ u_T^{(i)} \end{bmatrix} \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(0, \begin{bmatrix} \sigma_u^2 \Lambda_C(A, B, T) + \sigma_w^2 \Lambda_C(A, I, T) & 0 \\ 0 & \sigma_u^2 I_{n_u} \end{bmatrix}\right)
$$

where, $\Lambda_C(A, B, T) = \sum_{t=0}^{T} A^t B B^\top (A^\top)^t$ is the T-step controllability Grammian. Note that the singular values of the grammian relates to how easy the system is to identify.

Now we derive the bounds on $\|\hat{A} - A\|_2$, Define $Q_A = [I_{n_x} \; 0_{n_x \times n_u}]$, then

$$\begin{aligned}
\|\hat{A} - A\| &= \|Q_A(Z_N^\top Z_N)^{-1} Z_N^\top W_N\| \\
&= \|Q_A(\Sigma_x^{1/2} Y_N^\top \Sigma_x^{1/2})^{-1} \Sigma_x^{1/2} Y_N^\top W_N\| \\
&= \|Q_A \Sigma_x^{-1/2} (Y_N^\top Y_N)^{-1} Y_N^\top W_N\| \\
&= \|[\Sigma_x^{-1/2} \quad 0](Y_N^\top Y_N)^{-1} Y_N^\top W_N\|
\end{aligned}$$

(via the sub-multiplicative property of the spectral norm)

$$\leq \|\Sigma_x^{-1/2}\| \frac{\|Y_N^\top W_N\|}{\lambda_{min}(Y_N^\top Y_N)} = \lambda_{min}^{-1/2}(\Sigma_x) \frac{\|Y_N^\top W_N\|}{\lambda_{min}(Y_N^\top Y_N)}$$

where $Y_N := [y_i^\top]_{i=1}^N$, with $y_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, I_{n_x+n_u})$

In a similar manner, it can be shown that

$$\|\hat{B} - B\|_w \leq \frac{1}{\sigma_u} \frac{\|Y_N^\top W_N\|_2}{\lambda_{min}(Y_N^\top Y_N)}$$

5 Proof Strategy

In order to derive the bounds, similar to the scalar case:

- First, find high probability upper bound on $\|Y_N^\top W_N\|_2$
- Next, find high probability lower bound on $\lambda_{min}(Y_N^\top Y_N)$
- Union bound to combine the two bounds, similar to before.
- Need only one other trick to use scalar random variable concentration bounds to control the singular values of random matrices
- Start with upper bound
- Use the variational form of operator norm, pointwise bound, plus a covering argument

6 Upper bound on $\|Y_N^\top W_N\|_2$

First, we state the result,

Proposition 3. *Let $x_i \in \mathbb{R}^n$ and $w_i \in \mathbb{R}^m$ be such that $x_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma_x)$ and $w_i \overset{i.i.d.}{\sim} N(0, \Sigma_w)$, and let $M = \sum_{i=1}^N x_i w_i^\top$. Fixing a failure probability $\delta \in (0, 1]$ and let $N \geq \frac{1}{2}(n+m)\log(9/\delta)$. Then, with probability at least $1 - \delta$*

$$\|M\| \leq 4\|\Sigma_x\|^{1/2}\|\Sigma_w\|^{1/2}\sqrt{N(n+m)\log(9/\delta)}$$

Note how the bound depends on $n + m$, meaning it depends on the size of the system.

Proof. Define

$$M = \Sigma_x^{1/2}(\Sigma_{i=1}^N Y_i Z_i^T)\Sigma_w^{1/2} \qquad \text{where } Y_i \sim N(0, I_n), Z_i \sim N(0, I_m)$$

$$\|M\| \leq \|\Sigma_x^{1/2}\|\|\Sigma_w^{1/2}\|\|\Sigma_{i=1}^N Y_i Z_i^T)\|$$

Here,

$$\|\Sigma_{i=1}^{N} Y_i Z_i^T)\|_2 = \sup \sum_{i=1}^{N} (u^\top y_i)(z_i^\top v) \qquad \text{where } \|u\| = \|v\| = 1,\ u \in \mathbb{R}^n,\ v \in \mathbb{R}^m$$

It's at this point that we approximate the supremum with an ϵ-net.

Definition 1 (ϵ-net)**.** *(HDP, Ch. 4.2) Let (T, d) be a metric space. Consider a subset $K \subset T$ and let $\epsilon > 0$. A subset $\mathcal{N} \subseteq K$ is called an* ϵ-net of K *if every point in K is within distance ϵ of some point $\mathcal{N}$, i.e.*

$$\forall x \in K \exists x_0 \in \mathcal{N} : d(x, x_0) \leq \epsilon$$

Equivalently, $\mathcal{N}$ is an ϵ-net of K if and only if K can be covered by balls with centers in N and radii ϵ.

Definition 2 (Covering numbers)**.** *The smallest possible cardinality of an ϵ-net of K is called the* covering number *of K and is denoted $\mathcal{N}(K, d, \epsilon)$. Equivalently, $\mathcal{N}(K, d, \epsilon)$ is the smallest number of closed balls with centers in K and radii ϵ whose union covers K.*

Using the ϵ-net trick which grids up the space,

Let $\{u_k\}_{k=1}^{N_\epsilon}, \{v_k\}_{k=1}^{M_\epsilon}$ be such that,

$$\forall \|u\| = 1, \|u - u_k\| \leq \epsilon \qquad \text{for some } u_k$$
$$\forall \|v\| = 1, \|v - v_k\| \leq \epsilon \qquad \text{for some } v_k$$

In other words they are the ϵ-coverings of the $\mathcal{S}^{n-1}$ and $\mathcal{S}^{m-1}$, respectively. Then,

$$\begin{aligned} u^\top M v &= (u - u_k^\top) M v + u_k^\top M (v - v_l) + u_k^\top M v_l \\ &\leq \|u - u_k\| \|M\| \|v\| + \|u_k\| \|M\| \|v - v_l\| + u_k^\top M v_l \\ &= 2\epsilon \|M\| + \max_{k,l} u_k^\top M v_l \end{aligned}$$

Taking the supremum,

$$\|M\| \leq 2\epsilon \|M\| + \max_{k,l} u_k^\top M v_l$$
$$\|M\| \leq \frac{1}{1 - 2\epsilon} \max_{k,l} u_k^\top M v_l$$

If we set $\epsilon = 1/4$ and do a standard volume comparison, then we can show that

$$N_\epsilon \leq 9^m$$
$$M_\epsilon \leq 9^n$$

So the total number of pairs $(u_k, v_k) \leq 9^{n+m}$ Substituting back in, we get

$$\|\Sigma_{i=1}^{N} Y_i Z_i^T)\|_2 \leq 2 \max_{1 \leq k \leq 9^n, 1 \leq l \leq 9^m} \sum_{i=1}^{N} (w_k^\top y_i)(z_i^\top v_l)$$
$$\|\Sigma_{i=1}^{N} Y_i Z_i^T)\|_2 \leq 2 \sum_{i=1}^{N} (u_k^\top y_i)(z_i^\top v_l) \qquad \forall 9^{(m+n)}\ (u_k, v_l) \text{ pairs}$$

Applying the sum of sub-exponential r.v.s concentration bounds with probability of failure $\frac{\delta}{9^{n+m}}$. Then with probability $1 - \frac{\delta}{9^{n+m}}$

$$\sum (u_k^\top y_i)(z_t^\top v_l) \leq 2\sqrt{N \log(\frac{9^{n+m}}{\delta})} \leq 2\sqrt{N(n+m) \log \frac{1}{\delta}}$$

Union bound over all 9^{n+m} such events to get the result. Additional details can be found in [1] ■

7 Lower bound on $\lambda_{min}(Y_N^\top Y_N)$

Using a similar approach, we obtain the lower bound.

Proposition 4. *Let $x_i \in \mathbb{R}^n$ be drawn i.i.d. from $\mathcal{N}(0, \Sigma_x)$, and set $M = \sum_{i=1}^N x_i x_i^\top$. Fix a failure probability $\delta \in (0,1]$ and let $N \geq 24n\log(9/\delta)$. Then with probability at least $1-2\delta$,*

$$\lambda_{min}(M) \geq \lambda_{min}(\Sigma_x)N/2$$

Proof. The proof utilizes the previous results and an additional covering argument. Refer to [1] ■

8 Putting it all together

With Proposition 3 and Proposition 4, we union bound over all of the relevant failure probabilities.

Theorem 8.1. *Fix a failure probability $\delta \in (0,1]$ and assume that $N \geq 24(n_x+n_u)\log(54/\delta)$. Then it holds that with probability at least $1-\delta$,*

$$\|\hat{A} - A\|_2 \leq 8\sigma_w \lambda_{min}^{-1/2}(\Sigma_x)\sqrt{\frac{(2n_x+n_u)\log(54/\delta)}{N}}$$
$$\|\hat{B} - B\|_2 \leq 8\frac{\sigma_w}{\sigma_u}\sqrt{\frac{(2n_x+n_u)\log(54/\delta)}{N}}$$

Note, different bounds with better constants can be found in [2].

9 Data dependent bounds

The previous results rely on properties of the true underlying system and therefore cannot be used to implemented in practice. Therefore, two data-dependent approaches are used to compute error estimates.

Proposition 9.1. *Assuming we have N idependent samples $(y^{(l)}, x^{(l)}, u^{(l)})$ such that*

$$y^{(l)} = Ax^{(l)} + Bu^{(l)} + w^{(l)}),$$

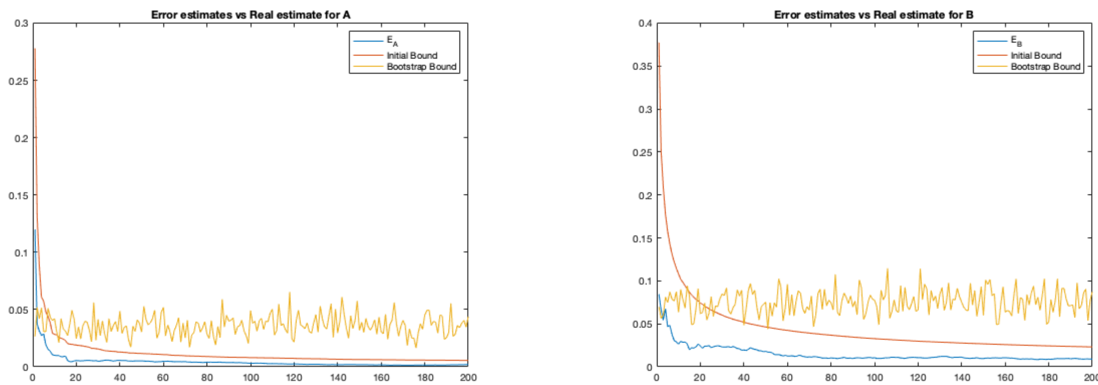
where $w^{(l)} \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2 I_{n_x})$ and are independent from $x^{(l)}$ and $u^{(l)}$. Assume that $N \geq n_x + n_u$, Then with probability $1-\delta$, we have

$$\begin{bmatrix}(\hat{A}-A)^\top \\ (\hat{B}-B)^\top\end{bmatrix}\begin{bmatrix}(\hat{A}-A)^\top \\ (\hat{B}-B)^\top\end{bmatrix}^\top \preceq C^2_{n_x,n_u,\delta}\left(\sum_{l=1}^N \begin{bmatrix}x^{(l)} \\ u^{(l)}\end{bmatrix}\begin{bmatrix}x^{(l)} \\ u^{(l)}\end{bmatrix}^\top\right)^{-1}$$

, where $C^2_{n_x,n_u,\delta} = \delta_w^2(\sqrt{n_x+n_u} + \sqrt{n_x} + \sqrt{2\log(1/\delta)})^2$. If the right hand side has zero as an eigenvalue, we define the inverse of that eigenvalue to be infinity.

The proof can be found in [2].

We can obtain better bounds by utilizing the bootstrap technique [3]. Lecture notes regarding bootstrap can be found in [4]. The algorithm can be found in [1].



(a) Estimates for A. Includes actual error and the data-dependent bounds, as well as the bootstrap error estimates

(b) Estimates for B. Includes actual error and the data-dependent bounds, as well as the bootstrap error estimates

Figure 1: Results of numerical example.

10 Numerical example

In order to demonstrate the previous results, we apply these methods to estimating the parameters in a planar model for a quadrotor.

More complicated formulations of the dynamics of a quadrotor are possible, but we linearize the dynamics around the hover state for a quadrotor. The resulting state-space model can be expressed as follows (Assuming full observability):

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$

where,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} r \\ p \\ t \end{bmatrix} A_k = \begin{bmatrix} 1 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} B_k = \begin{bmatrix} 0.049 & 0 & 0 \\ 0 & 0.049 & 0 \\ 0 & 0 & 0.01 \\ 0.98 & 0 & 0 \\ 0 & -0.98 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

We set the parameters, gravity $g = 9.8m/s^2$, the mass of the quadrotor $m = 0.5kg$, the inputs are bounded with roll r, pitch $p \leq 45$ degrees, and thrust $t \leq 5$. The matrices shown are after discretization.

For the experiment, $\sigma_w = 0.1$, $\sigma_{u,r} = \sigma_{u,p} = 0.7854/3$, $\sigma_{u,t} = 5/3$.

The results are shown in Fig. 1a and 1b.

As can be seen from the figure, the data-dependent bounds estimate the overall error across the number of rollouts and improves as the number of rollouts increases. The bootstrap bound *also shows improvement* (Actually did not work).

References

[1] Nikolai Matni and Stephen Tu. A Tutorial on Concentration Bounds for System Identification. *arXiv:1906.11395 [cs, math, stat]*, August 2019. arXiv: 1906.11395.

[2] Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. On the Sample Complexity of the Linear Quadratic Regulator. *arXiv:1710.01688 [cs, math, stat]*, October 2017. arXiv: 1710.01688.

[3] B. Efron. Bootstrap Methods: Another Look at the Jackknife. *The Annals of Statistics*, 7(1):1–26, January 1979.

[4] Lecture13.pdf.

The (very naive and inefficient) code used to derive the results are shown here:
The code for this problem is as follows:

```
clear;
clc;
close all;
%Get discretized dynamics
PlantModelQuadSimpleLinear;

rpmax = deg2rad(45); %max roll pitch angle
thrmax = 5*1; %max thrust

%Collect N trials (rollouts), horizon T, find A_hat, B_hat

N = 200;
T = 50;
x0= zeros(6,1);
initial_data_xt  = [];
initial_data_xt_1 = [];
std_u_rp = rpmax/3;
std_u_t = thrmax/3;
sigma_w = 0.1;
fail_prob = 0.01;

big_A = [sys_d.A zeros(6,3);zeros(6,6) sys_d.B];
E_A = zeros(N,1);
E_B = zeros(N,1);
bound_A_init = zeros(N,1);
bound_B_init = zeros(N,1);
bootstrap_eA = zeros(N,1);
bootstrap_eB = zeros(N,1);


for n_ind  = 1:N
    temp_x = zeros(T,9);
    temp_y = zeros(T,6);
    u0 = [std_u_rp*randn(2,1); std_u_t*randn(1,1)];
    z0= [x0 ; u0];
    x_t = x0;
    u_t = u0;
    z_t = z0;
    for t_ind = 1:T
        x_t_1 = sys_d.A*x_t +sys_d.B*u_t + sigma_w*randn(6,1);
        temp_y(t_ind, :) = x_t_1';
        temp_x(t_ind, :) = [x_t;u_t]';

        u_t = [std_u_rp*randn(2,1); std_u_t*randn(1,1)];
        x_t = x_t_1;
    end
    initial_data_xt = [initial_data_xt; temp_x];
    initial_data_xt_1 = [initial_data_xt_1; temp_y];


    beta = mvregress(initial_data_xt,initial_data_xt_1);
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A_hat  = beta(1:6, :)';
B_hat  = beta(7:9, :)';


%Compute initial error
E_A(n_ind) = norm(A_hat - sys_d.A);
E_B(n_ind) = norm(B_hat - sys_d.B);

%Compute intial bounds
C_squared = sigma_w^2*( sqrt(9)+ sqrt(6)+ sqrt(2*log(1/fail_prob)))^2;
error= [(A_hat - sys_d.A)'; (B_hat - sys_d.B)'];
error_norm = error*error';
M_mat = inv(initial_data_xt' * initial_data_xt);
error_bound_matrix = C_squared*M_mat;

%Compute bound for A, B
QA = [eye(6) zeros(6,3)];
QB = [zeros(3,6) eye(3)];

bound_A_init(n_ind) = sqrt(C_squared)*sqrt(norm(QA*M_mat*QA'));
bound_B_init(n_ind) = sqrt(C_squared)*sqrt(norm(QB*M_mat*QB'));


fprintf('Rollout %d\n',n_ind)
fprintf('Initial E_A: %g\n', E_A(n_ind))
fprintf('Initial Bound: %g\n', bound_A_init(n_ind))
fprintf('Initial E_B: %g\n', E_B(n_ind))
fprintf('Initial Bound: %g\n', bound_B_init(n_ind))

%Boostrap estimation of E_A and E_B, M times

M= 100;
L = 50;
bootstrap_data_xt = [];
bootstrap_data_xt_1 = [];
bootstrap_eA_set = zeros(M,1);
bootstrap_eB_set = zeros(M,1);
for m_ind  = 1:M
    for l_ind = 1:L
        temp_x = zeros(T,9);
        temp_y = zeros(T,6);
        u0 = [std_u_rp*randn(2,1); std_u_t*randn(1,1)];
        z0= [x0 ; u0];
        x_t = x0;
        u_t = u0;
        z_t = z0;
        for t_ind = 1:T
            x_t_1 = A_hat*x_t +B_hat*u_t + sigma_w*randn(6,1);
            temp_y(t_ind, :) = x_t_1';
            temp_x(t_ind, :) = [x_t;u_t]';

            u_t = [std_u_rp*randn(2,1); std_u_t*randn(1,1)];
            x_t = x_t_1;
        end
        bootstrap_data_xt = [bootstrap_data_xt; temp_x];
        bootstrap_data_xt_1 = [bootstrap_data_xt_1; temp_y];
    end

    %Find A_tilde, B_tilde

    beta = mvregress(bootstrap_data_xt,bootstrap_data_xt_1);
    A_tilde = beta(1:6, :)';
    B_tilde  = beta(7:9, :)';
    bootstrap_eA_set(m_ind) = norm(A_tilde - A_hat);
    bootstrap_eB_set(m_ind) = norm(B_tilde - B_hat);
```

```
    end

    %output 100(1-delta) percentile
    bootstrap_eA(n_ind) = quantile(bootstrap_eA_set, 1-fail_prob);
    bootstrap_eB(n_ind) = quantile(bootstrap_eB_set, 1-fail_prob);

    fprintf('Bootstrap E_A: %g\n', bootstrap_eA(n_ind))
    fprintf('Bootstrap E_B: %g\n', bootstrap_eB(n_ind))

end

save ('results.mat')

figure; plot(E_A); hold on; plot(bound_A_init); plot(bootstrap_eA);
legend({'E_A'; 'Initial Bound'; 'Bootstrap Bound'})
title('Error estimates vs iterations for A')
saveas(gcf, 'E_A_plot2.png')

figure; plot(E_B); hold on; plot(bound_B_init); plot(bootstrap_eB);
legend({'E_B'; 'Initial Bound'; 'Bootstrap Bound'})
title('Error estimates vs iterations for B')
saveas(gcf, 'E_B_plot2.png')
```