ESE 605-001: Modern Convex Optimization **Spring 2021**

Homework 1

Assigned: 01/26/2021 *Due: 02/12/2021*

Homework must be LaTeX'd or it will not be graded.

You are expected to provide complete and rigorous solutions to all questions. Do not leave portions of your solutions as "exercises for the grader."

Problems from Boyd & Vandenberghe: 2.8, 2.12, 2.15 (a-f), 2.19, 2.22, 2.31

Problems from additional exercises
`https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook_extra_exercises.pdf:` A1.6

Problems from Boyd & Vandenberghe: 3.4, 3.8, 3.13, 3.14

Problems from additional exercises
`https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook_extra_exercises.pdf:` A2.2

Bonus questions: Bonus questions are completely optional. If a certain threshold of correctness is exceeded, you will earn up to an additional 2 marks per question on your assignment grade. These are fun, challenging problems, and we ask that you try your best to get as far into the proofs/answers as you can *without consulting any outside sources!* Once you get stuck, indicate the point at which you were stuck in your solutions with a "**I made it this far on my own**," after which, you should indicate what outside source you consulted in order to finish the problem, in accordance with the Penn Academic Integrity policy.

1. Describe in simple terms the convex hull of the set of special orthogonal matrices in $\mathbb{R}^3$:

$$SO(3) = \{U \in \mathbb{R}^{3\times 3} \,|\, U^\top U = I, \, \det U = 1\}.$$

2. Prove an approximate version of Caratheodory's theorem: Let $S \subset \mathbb{R}^d$ be a set satisfying $\sup_{x,y\in S} \|x - y\|_2 \le 1$. Then, for every point $x \in \operatorname{conv}(S)$ and every integer k, one can find points $x_1, \ldots, x_k \in S$ such that

$$\left\| x - \frac{1}{k}\sum_{j=1}^{k} x_j \right\|_2 \le \frac{1}{\sqrt{k}}.$$

Note two perhaps shocking features of this result: (i) the number of points k in the convex combination needed for a $1/\sqrt{k}$ approximation does not depend on the ambient dimension n, and (ii) the coefficients of the convex combination can all be made equal (but note that we do allow for repetitions among the points x_k).

3. For $x \in \mathbb{R}^d$ define $M^k(x) := \sum_{j=1}^{k} x_{[j]}$ to be the max $-k$-sum, where $x_{[1]}, \ldots, x_{[d]}$ are the elements of x sorted in non-increasing order. Show that $M^k(x)$ is convex by showing that its *epigraph* admits a compact convex representation in terms of a small number of affine inequalities.

4. Prove the following: for a given A, b, exactly one of the following statements is true:

   (a) there exists an x with $Ax = b$, $x \ge 0$;

   (b) there exists a y with $A^\top y \ge 0$, $b^\top y < 0$